Functional Inequalities for Markov semigroups

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Abstract : In these notes, we describe some of the most interesting inequalities related to MARKOV semigroups, namely spectral gap inequalities, Logarithmic Sobolev inequalities and Sobolev inequalities. We show different aspects of their meanings and applications, and then describe some tools used to establish them in various situations.

Contents

Introduction

The analysis of MARKOV semigroups is related to the study of second order subelliptic differential operators and of Dirichlet forms on one side, and to the study of MARKOV processes on the other side. Among the main questions about them is their behaviour in small or large time, in particular the study of convergence to equilibrium and the control of it. To study such questions, it is interesting to look at some functional inequalities related to these semigroups, which are in general inequalities relating L^p-norms of functions to L^q norms of their gradients. There is a full zoology of such inequalities, and we chose to restrict ourselves in this course to the simplest and the most important ones (at least for our point of view), which are spectral gap inequalities, logarithmic Sobolev inequalities and Sobolev inequalities. After a general presentation of MARKOV semigroups, we describe these main inequalities and how to use them in different contexts. Then, we show how the study of the local structure of generators of semigroups may be used to prove existence of these inequalities. The spectral gap and logarithmic SOBOLEV inequalities are quite easy to handle, whereas the study of SOBOLEV inequalities is much more difficult.

This course is divided into five chapters : in the first one, we give a general presentation of MARKOV semigroups, with the main examples we have in mind. In the second chapter, we present the three fundamental inequalities and show how they are related to the behaviour of the corresponding semigroup, in small and large time, and how they are related to integrability properties of LIPSCHITZ functions. In Chapter 3, we introduce the curvature-dimension inequalities and relate them to different functional inequalities. Chapters 4 and 5 are more concentrated on the SOBOLEV inequalities. In Chapter 4, we underline the conformal invariance of the SOBOLEV inequalities, and show the use which may be made of this invariance. Then, in Chapter 5, we introduce some non-linear evolution equations, related to the porous media equation, which may be used to obtain some SOBOLEV inequalities.

1 Markov semigroups.

1.1 Generators and invariant measures.

All the analysis described below takes place on a measure space $(E, \mathcal{E}, \mu)$, where μ is a non zero, σ-finite positive measure on the measurable space $(E, \mathcal{E})$. To avoid complications, we shall always assume that the σ-algebra $\mathcal{E}$ is generated by a denumerable family of sets up to sets of measure 0 (which means that there is a separable σ-algebra $\mathcal{E}_0$ whose μ-completion is $\mathcal{E}$). This covers all possibly imaginable cases one could have in mind when dealing with aforementioned questions. Throughout this paper, for $p \in [1, \infty)$, $\|f\|_p$ will denote the $L^p(\mu)$ norm of a function f:

$$\|f\|_p^p = \int |f|^p d\mu.$$

In the same way, we shall use the notation $\|f\|_\infty$ to denote the essential supremum with respect to μ of the function $|f|$.

A MARKOV semigroup $(P_t(x, dy))$ on E is a family of probability kernels on E

depending on the parameter $t \in \mathbb{R}_+$, such that, for any $t, s \in \mathbb{R}_+$,

$$\int_{y \in E} P_s(x, dy) P_t(y, dz) = P_{s+t}(x, dz). \tag{1}$$

(This is called the CHAPMAN-KOLMOGOROV identity.) It is entirely characterized by it's action on positive or bounded measurable functions by

$$P_t f(x) = \int f(y)\, P_t(x, dy).$$

The family of operators P_t therefore satisfies

$$P_t \circ P_s = P_{t+s},\ P_t 1 = 1,\ f \geq 0 \Longrightarrow P_t f \geq 0.$$

Moreover, we shall require that, for any function f in $L^2(\mu)$,

$$\lim_{t \to 0^+} P_t f = f,$$

the above limit being taken in $L^2(\mu)$.

MARKOV semigroups appear naturally in the study of MARKOV processes, where the probability measure $P_t(x, dy)$ is the law of a MARKOV process (X_t) starting from the point x at time 0 (see Section 1.4 below).

Observe that, since P_t is given by a semigroup of probability measures, for any convex function ϕ such that the two sides make sense, we have

$$P_t \phi(f) \geq \phi(P_t f), \tag{2}$$

by JENSEN's inequality.

The relationship between the measure μ and the semigroup P_t is the following : we shall require μ to be an invariant measure for P_t, namely that, for any positive function $f \in L^1(\mu)$,

$$\int P_t f(x) \mu(dx) = \int f(x) \mu(dx).$$

(Sometimes, we shall require a bit more, the symmetry property of the measure, see section 1.2 below.) In most of the cases, given a semigroup P_t, such a measure shall exist and be unique, up to a multiplicative constant. If the measure is finite, we shall always normalize it to a probability measure, i.e. $\mu(E) = 1$.

If μ is invariant, then it is clear that P_t is a contraction in $L^1(\mu)$ for any t. Since it is also a contraction in $L^\infty(\mu)$, by interpolation it is a contraction in $L^p(\mu)$, for any $p \in [1, \infty]$.

One of the main questions addressed by these lectures is to obtain the convergence of $P_t f$ to $\int f d\mu$ when t goes to infinity, and to control this convergence, at least when μ is a probability measure.

Since P_t is a semigroup of bounded operators in any $L^p(\mu)$, one may apply the Hille-Yoshida theory ([41]): for $p \in [1, \infty)$, the generator

$$L(f) = \lim_{t \to 0^+} \frac{1}{t}(P_t f - f)$$

exists in $L^p(\mu)$ on a dense subspace $\mathcal{D}_p$ of $L^p(\mu)$, and the description of L and of $\mathcal{D}_p$ entirely characterizes the semigroup P_t by the fact that, given $f \in \mathcal{D}_p$, the function $F(x,t) = P_t f(x)$ is the unique solution of the heat equation

$$\frac{\partial}{\partial t} F(x,t) = LF(x,t)$$

which is in $\mathcal{D}_p$ for any $t > 0$, the derivative being taken in $L^p(\mu)$. Moreover, we have in $L^p(\mu)$

$$\frac{\partial}{\partial t} P_t = LP_t = P_t L. \tag{3}$$

In general, it is not an easy task to describe the domain $\mathcal{D}_p$. What shall be given is a subspace $\mathcal{A}$ of the domain, dense in the domain topology described by the norm $\|f\|_{\mathcal{D}_p} = \|f\|_p + \|Lf\|_p$. In order to make things simpler, we shall make in what follows the following assumption : $\mathcal{A}$ is an algebra, stable under L and P_t, and stable under composition with smooth functions with value 0 at 0, and having at most polynomial growth at infinity together with all their derivatives. Moreover, when the measure μ is finite, we shall require constant functions to belong to $\mathcal{A}$.

This very strong assumption is made here to justify in all circumstances the computations present in these notes. The fact that $\mathcal{A}$ is an algebra is already quite restrictive. For example, it rules out the case of fractal diffusions, which in general do not satisfy this hypothesis (they are not of LEBESGUE type). The most unreasonable assumption is the stability under P_t. This formally shall restrict ourselves to very particular settings : for example, in the diffusion case on manifolds described below, one would like to take for $\mathcal{A}$ the set of compactly supported smooth functions, but then it is never stable under P_t unless the manifold is compact. But this hypothesis is absolutely not necessary, and may be replaced by more technical assumptions in practice (see [5], p. 24, for example). Nevertheless, we shall stick to this hypothesis since our aim in these lectures is to present simple ideas which may be used in a wider context, provided one adapts the arguments to the many different settings which could occur.

In any case, we shall use these properties of $\mathcal{A}$ only in Chapter 3 and beyond. For the beginning, the reader may assume that $\mathcal{A}$ is just the L^2 domain of the operator L.

In terms of L, the invariant measure μ may be described to be any positive solution of the equation $L^*(\mu) = 0$, L^* being the adjoint of the operator L acting on measures. In other words, one has

$$\int L(f) d\mu = 0 \tag{4}$$

for any $f \in \mathcal{A}$. It is not easy to describe general conditions on L for such a solution to be unique. For example, in statistical mechanics, one constructs the operator L in such a way that all the solutions are GIBBS' measures, and unicity characterizes the absence of phase transition.

Notice that the inequality (2) together with (3) considered at $t = 0$ immediately leads to

$$L(\phi(f)) \geq \phi'(f) Lf, \tag{5}$$

valid for any convex function ϕ and $f \in \mathcal{A}$.

This inequality is characteristic for generator of MARKOV semigroups. In fact, if ϕ is convex and μ invariant, one has

$$\frac{\partial}{\partial t}\int \phi(P_tf)\,d\mu = \int \phi'(P_tf)LP_tf\,d\mu \leq \int L[\phi(P_tf)]\,d\mu = 0.$$

Therefore,

$$\int \phi(P_tf)\,d\mu \leq \int \phi(f)d\mu.$$

Then, if we apply this to a non-negative function f with $\phi(x) = |x|$, we get

$$\int |P_tf|\,d\mu \leq \int f\,d\mu = \int P_tf\,d\mu,$$

and therefore P_tf is again non-negative. Therefore, inequality (5) characterizes the positivity preserving property of P_t. Moreover, if $L(1) = 0$, then P_t1 is constant and $P_t1 = 1$.

On $\mathcal{A}$, one can now define the "carré du champ" operator (squared field operator), which is the symmetric bilinear map from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ defined as

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - fL(g) - gL(f)). \tag{6}$$

From (5), applied with $\phi(x) = x^2$, it is clear that $\Gamma(f,f) \geq 0$ for all $f \in \mathcal{A}$. Moreover, if the constant function 1 is in $\mathcal{A}$, then $L(1) = 0$ and $\Gamma(f,1) = 0$, for every $f \in \mathcal{A}$.

In fact, it can be proved that if a bilinear operator Γ is constructed as above from a linear operator L on a commutative algebra $\mathcal{A}$, and if $\Gamma(f,f)$ is positive for all $f \in \mathcal{A}$, then for every convex polynomial ϕ, $L(\phi(f)) \geq \phi'(f)L(f)$ (see [31]). Aproximating in an appropriate way the function $x \mapsto |x|$ by convex polynomials, one sees that it is in fact the positivity of Γ which carries the positivity property of P_t (see [31] for more details.)

Furthermore, let us observe that

$$\int \Gamma(f,f)\,d\mu = -\int fL(f)\,d\mu, \tag{7}$$

equation which will be used throughout these notes.

1.2 Symmetry.

One the two basic properties that may be or not be shared by the semigroup is the symmetry (or reversibility). It asserts that operators P_t are symmetric in $L^2(\mu)$, or equivalently that the operator L is self-adjoint on it's domain in $L^2(\mu)$. For unbounded operators, self-ajdointness is a stronger property than symmetry, since it requires that the domain is also the domain of the adjoint operator, or, in other words, that the domain is maximal. This is always the case for generators of symmetric semigroups of bounded operators on an Hilbert space (see [41], e.g.) Since $\mathcal{A}$ is dense in the domain topology, the selfadjoint property is equivalent to the symmetry property on $\mathcal{A}$. Symmetry is then formulated as

$$\forall f,g \in L^2(\mu), \qquad \int fP_tg\,d\mu = \int gP_tf\,d\mu, \tag{8}$$

or, in an equivalent form,

$$\forall f, g \in \mathcal{A}, \int fL(g)\, d\mu = \int gL(f)\, d\mu. \tag{9}$$

Notice that this property implies that μ is reversible (take $g = 1$ in (9)). Therefore, since in most cases μ is unique and entirely determined from L, this is a particular class of semigroups which are symmetric with respect to their invariant measures. We shall see that with more details on the examples of Section 1.7 below.

In this case, L has a spectral decomposition in $L^2(\mu)$. Because of the equation (7), one has $\int fLf\, d\mu \leq 0$, and the spectrum of $-L$ is included in $[0, \infty)$. This spectral decomposition may therefore be written as

$$L = -\int_0^\infty \lambda dE_\lambda,$$

in which case

$$P_t = \int_0^\infty e^{-\lambda t} dE_\lambda.$$

Then, we have $P_t = \exp(tL)$, equation that we shall write even in the non-symmetric case where this expression has to be justified more carefully.

Moreover, in this case, the operator L (and therefore the corresponding semigroup) is entirely determined by μ and Γ, since then

$$\int gLf\, d\mu = -\int \Gamma(f, g)\, d\mu.$$

1.3 Diffusions.

The second basic property is the diffusion property. In an abstract way, it asserts that L is a second order differential operator.

We may write it as follows : for all function $f \in \mathcal{A}$ and all smooth function $\phi : \mathbb{R} \mapsto \mathbb{R}$ such that $\phi(0) = 0$,

$$L(\phi(f)) = \phi'(f)Lf + \phi''(f)\Gamma(f, f). \tag{10}$$

With some elementary manipulations, it is easy to see that equation (10) implies

$$\forall (f, g, h) \in \mathcal{A}^3,\ \Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h), \tag{11}$$

which says that Γ is a first order differential operator in each of its arguments.

From equation (11), it is easy to see that the identity (10) is valid, at least when ϕ is a polynomial. The extension of the identity (10) from polynomials to general smooth functions ϕ requires some analysis (and certainly more hypotheses on $\mathcal{A}$ than the ones we made).

This abstract formulation in terms of the algebra $\mathcal{A}$ allows us to consider diffusions on general measurable spaces E, without refering to any differential structure or even any topology on E: it is the fact that the functions in $\mathcal{A}$ are in the domain of L^k, for any k, and therefore "smooth" in any reasonnable sense, which determines the differentiable structure. Of course, in concrete exemples, E shall very often be a smooth manifold, but we may consider diffusion semigroups on the Wiener space (the space of continuous functions on the unit interval, see Section 1.7.4 below), or, in statistical mechanics, on infinite products of smooth manifolds .

Of course, there is a multivariable analog of this chain rule formula which is straightforward : for a smooth function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ and all $(f_1, \ldots, f_n) \in \mathcal{A}^n$,

$$L\phi(f_1, \ldots, f_n) = \sum_i \frac{\partial \phi}{\partial x_i}(f_1, \ldots, f_n) L(f_i) + \sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j}(f_1, \ldots, f_n) \Gamma(f_i, f_j). \tag{12}$$

The reason why we restrict ourselves to second order differential operators is that, given any differential operator L on a smooth manifold, if its associated Γ operator is positive, then it has to be of second order.

This diffusion property may never hold on a discrete space : if $f \in \mathcal{A}$ and if the diffusion property holds, then the image measure of μ through f has a connected support (intermediate value theorem, see [11]).

As we shall see later on examples, in the diffusion symmetric case, the operator Γ encodes the second order part of the operator L, while the measure μ describes the first order part. But this description is not really relevant for non-diffusion semigroups.

1.4 Probabilistic interpretation.

Markov semigroups appear naturally in the study of Markov processes, where

$$P_t f(x) = \mathbf{E}_x(f(X_t)), \tag{13}$$

(X_t) being a Markov process with initial value x at time $t = 0$. In other words, $P_t(x, dy)$ is the law of the random variable (X_t) when the initial value of the process is the point x. The Chapman-Kolmogorov equation (1) is then the translation of the Markov property of the process (X_t).

Then, we may chose a version of it such that, for $f \in \mathcal{A}$, the real processes $f(X_t)$ are right continuous with left limits (càdlàg, in short and in french). The link with the generator L is made through the martingale problem associated with it : for any $f \in \mathcal{A}$, the process $f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$ is a local martingale.

The diffusion property is then seen through the fact that the processes $f(X_t)$ must have continuous trajectories for any $f \in \mathcal{A}$ (see [10]).

1.5 Natural distance.

As we shall see later, the operator $\Gamma(f, f)$ stands for the square of the lenght of the gradient of a function f. It is natural to think that a Lipschitz function is a function for which $\Gamma(f, f)$ is bounded. Therefore, there is a natural distance associated to the operator Γ which can be written as

$$d(x, y) = \sup_{\Gamma(f,f) \le 1} (f(x) - f(y)).$$

This distance has been introduced in [15, 16]. It may well happen that this distance is almost everywhere infinite, but in reasonable situations it is well defined and carries a lot of information about the structure of the semigroup P_t. If our functions in $\mathcal{A}$

are only defined almost everywhere, a more natural distance between sets of positive measures may be defined as

$$d(A,B) = \sup_{\Gamma(f,f)\le 1}\{\operatorname*{essinf}_A(f) - \operatorname*{essup}_B(f)\}.$$

This notion of distance is more adapted to infinite dimensional settings, and has been used in [27, 28] for example.

1.6 Tensorization

When we have two semigroups $P_t^1(x_1, dy_1)$ and $P_t^2(x_2, dy_2)$ acting on (E_1, μ_1) and (E_2, μ_2) separately, we may consider the semigroup $P_t = P_t^1(x_1, dy_1) \otimes P_t^2(x_2, dy_2)$ acting on the product space $(E_1 \times E_2, \mu_1 \otimes \mu_2)$. This is again a MARKOV semigroup with invariant measure $\mu_1 \otimes \mu_2$. We denote it's generator by $L_1 + L_2$ and it's carré du champ by $\Gamma_1 + \Gamma_2$. The corresponding algebra $\mathcal{A}$ may be chosen as the tensor product of the two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ corresponding to the two semigroups (P_t^1) and (P_t^2).

The notation $L_1 + L_2$ has to be understood in the following way : if $f(x_1, x_2)$ is a function of the two variables, then L_1 acts on the variable x_1, x_2 being fixed, and symmetrically for L_2. The same holds for the operator Γ. The corresponding MARKOV process on the product space $E_1 \times E_2$ is then simply two independent MARKOV processes on each of the coordinates.

This construction may also be made by taking infinite products, provided the corresponding measures are probability measures. In particular, it is sometimes interesting to consider $(P_t^{\otimes \mathbb{N}})$ on $(E^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$, provided that μ is a probability measure. We shall see an example of this situation in Section 1.7.4.

1.7 Basic examples.

1.7.1 Finite sets.

The first basic example is related to MARKOV processes on a finite set E. In that case, the operator L is given by a matrix $(L(x,y))$, $(x,y) \in E^2$, and $Lf(x) = \sum_y L(x,y)f(y)$. The matrix L must satisfy $L(x,y) \ge 0$ when $x \ne y$ and, for any $x \in E$, $\sum_{y\in E} L(x,y) = 0$.

The second condition asserts that $L(1) = 0$, and the first gives the positivity of Γ, since in this case

$$\Gamma(f,g)(x) = \frac{1}{2}\sum_y L(x,y)(f(x) - f(y))(g(x) - g(y)).$$

The semigroup P_t is given by the matrix $P_t(x,y) = (\exp(tL))(x,y)$, and it is an elementary exercise to check that the exponential of a matrix L has non negative entries if and only if the matrix L has off-diagonal non negative entries. The measure μ must then satisfy

$$\forall y \in E,\ \mu(y) = \sum_{x\in E} \mu(x)L(x,y).$$

There always exists a probability measure satisfying this equation. It is well known that this probability measure is unique if and only if the matrix L has only one recurrence class. It is then carried by the recurrent points, and this uniqueness is equivalent to the fact that any function with $\Gamma(f,f)=0$ must be constant on the set of recurrent points.

The measure is reversible if and only if it satisfies

$$\forall (x,y)\in E^2,\ \mu(x)L(x,y)=\mu(y)L(y,x),$$

which is known in the MARKOV chain literature as the detailed balance condition.

1.7.2 Compact manifolds.

The second basic example is the case where E is a compact connected smooth manifold, and the operator L is given in a system of local coordinates by

$$Lf(x)=\sum_{i,j} g^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}+\sum_i b^i(x)\frac{\partial f}{\partial x^i},$$

where $(g^{ij})(x)$ is a symmetric smooth matrix which is positive definite, and $b^i(x)$ is a smooth set of coefficients (in this form, it is not a vector field since it does not follow the usual rules under a change of variables).

Since the matrix $(g^{ij})(x)$ is positive definite at each point, the inverse matrix $(g_{ij})(x)$ defines a Riemannian metric. The LAPLACE-BELTRAMI operator of this metric has the same second order terms as L, and therefore L may be written in a canonical way as $L=\Delta_g+X$, where Δ_g is the LAPLACE-BELTRAMI operator of the metric (g) and X is a smooth vector field (which means a first order operator with no 0-order term). Then the algebra $\mathcal{A}$ may be taken as the algebra of smooth functions on E. The square field operator is

$$\Gamma(f,g)=\sum_{i,j} g^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}=\nabla f.\nabla g,$$

and therefore $\Gamma(f,f)=|\nabla f|^2$, that is the square of the lenght of the vector field ∇f computed in the Riemannian metric. The natural distance associated with L is then the Riemannian distance.

In this case, when $X=0$, the RIEMANN measure, which in a local system of coordinates is

$$m(dx)=\sqrt{\det(g_{ij})}dx^1\dots dx^n$$

is the invariant measure, and is reversible. The measure μ is invariant if its density ρ with respect to the RIEMANN measure m satisfies the equation

$$\Delta(\rho)-X(\rho)-\mathrm{div}(X)\rho=0.$$

The measure μ is reversible if and only if $X=\nabla h$, in which case $\rho=\exp(h)$.

One may also often consider the case of non compact manifolds. Then, when the operator may be written in the form $\Delta+\nabla h$, the algebra of smooth compactly supported functions is dense in the domain as soon as the manifold is complete in the Riemannian metric, which amounts to say that there exists a sequence of functions (f_n) in $\mathcal{A}$ such that $0\le f_n\le 1$, (f_n) increasing to 1, and such that $\Gamma(f_n,f_n)$ converges uniformly to 0. This algebra is not stable under P_t in general, but a precise analysis shows that all the methods described below still work in this context. In this situation, the associated semigroup is not always Markov, and we just have in general $P_t 1 \leq 1$ (sub- Markov property). In order to ensure that the semigroup is Markov, it is enough to assume that the Ricci curvature associated to L (see the end of Section 3) is bounded from below (see [4]).

When the manifold is not complete, we have to deal in general with the boundary, and we systematically chose the Neuman boundary condition since we want the semigroup to be Markov. In this case, the algebra is the set of smooth functions with 0 normal derivative at the boundary, but only the case of boundaries which are convex with respect to the Riemann structure may be studied (with many technical difficulties) by the methods described below.

1.7.3 Orthogonal polynomials.

An important family of Markov semigroups are those associated to orthogonal polynomials on the real line. Namely, let us consider a probability measure μ on the real line, not supported by a finite set. Assume that μ has exponential moments, that is there exists some $\varepsilon > 0$ such that $\int e^{\varepsilon|x|}\mu(dx) < \infty$. Then the polynomials are dense in $L^2(\mu)$ and there exists a unique family of polynomials (Q_n) such that Q_n has degree n, that

$$\int Q_n Q_m \, d\mu = \delta_{m,n},$$

and that the leading coefficient of Q_n is positive. They are the orthogonal polynomials associated to μ. When are Q_n the eigenvectors of a Markov semigroup? More precisely, we are looking for those measures for which there exists a Markov semigroup P_t with generator L such that for some sequence (λ_n) of positive real numbers one has $LQ_n = -\lambda_n Q_n$ or equivalently $P_t Q_n = e^{-\lambda_n t} Q_n$. In this situation, the measure μ would be reversible for (P_t) and the algebra $\mathcal{A}$ may be chosen to be the algebra of polynomials (it which case it is not stable by composition with smooth functions but only with polynomials). Moreover, the semigroup $P_t(x, dy)$ may in this context be written as

$$P_t(x, dy) = \sum_n e^{-\lambda_n t} Q_n(x) Q_n(y) \mu(dy),$$

provided, of course, that for any t the series $\sum_n e^{-\lambda_n t}$ is convergent, which ensures that the previous series is convergent in $L^2(\mu \otimes \mu)$.

The answer in general is not known, but the problem is far simpler if we require L to be a diffusion operator. In this case, there are only three families, up to translations and dilations (see [30]) :

• The Jacobi polynomials : the measure μ has support $[-1, 1]$ and

$$\mu_{m,n}(dx) = (1+x)^{m/2-1}(1-x)^{n/2-1}\, dx/Z_{m,n},$$

where m and n are two real positive parameters and $Z_{m,n}$ is a normalizing constant. Then,

$$\lambda_k = k(k + \frac{m+n}{2} - 1),$$

and

$$L_{m,n}f(x) = (1-x^2)f''(x) - [\frac{n}{2}(x+1) + \frac{m}{2}(x-1)]f'(x),$$
$$\Gamma(f,f)(x) = (1-x^2)f'^2(x).$$

We shall call $L_{n,m}$ the Jacobi operator .

• The Hermite polynomials : the measure μ is the standard Gaussian measure on $\mathbb{R}$

$$\mu(dx) = e^{-x^2/2}\,dx/\sqrt{2\pi},\ \lambda_k = k$$
$$Lf(x) = f''(x) - xf'(x),\ \Gamma(f,f)(x) = f'^2(x).$$

This operator is known as the Ornstein-Uhlenbeck operator.

• The Laguerre polynomials : the measure μ is supported by the positive real line $[0,\infty)$, and

$$\mu(dx) = e^{-x}x^{n/2-1}\,dx/Z_n,$$

where n is a real positive number, Z_n is a normalizing constant, and

$$\lambda_k = k,\ Lf(x) = xf''(x) - (\frac{n}{2} - x)f',\ \Gamma(f,f)(x) = xf'^2(x).$$

For integer values of the parameters, those operators and semigroups have nice geometric interpretations. When $m = n$ is an integer, the Jacobi operator is related to the radial part of the Laplace-Beltrami operator of the unit sphere in $\mathbb{R}^{n+1}$ acting on radial (or zonal) functions. More precisely, consider the unit sphere S^n in $\mathbb{R}^{n+1}$, and it's Laplace-Beltrami operator Δ_{S^n}. Take a function $F(x)$ which depends only on the first coordinate in $\mathbb{R}^{n+1}$ (a zonal function), say $F(x) = f(x_1)$, and let Δ_{S^n} act on it. Then, one gets again a zonal function and

$$\Delta_{S^n}F(x) = L_{n,n}f(x_1).$$

Therefore, the operator $L_{n,n}$ may be seen as the 1-dimensionnal projection of Δ_{S^n}. Since zonal functions are also the functions which depend only on the Riemannian distance to the point $(1,0,\dots,0)$, $L_{n,n}$ is also the radial part of the Laplace-Beltrami operator of the sphere. Since the operator projects, is is quite clear that it's invariant measure projects too, and that the measure $\mu_{n,n}$ is the radial projection of the uniform measure of the sphere.

Now, if instead of projecting on a 1-dimensional axis, we project the sphere on a p-dimensional subspace of $\mathbb{R}^{n+1}$, with $p \leq n$, that is we let the operator Δ_{S^n} act on functions depending only on functions f depending only on $(x_1,\dots,x_p)$, then we get in the same way an operator $\Delta_{n,p}$ on the unit ball $\{\|x\| \leq 1\}$ of $\mathbb{R}^p$. This operator again has a radial symmetry, which means that if a function in the unit ball is radial, say $F(x) = f(\|x\|^2)$, then $\Delta_{n,p}F$ is again radial. We shall get $\Delta_{n,p}F(x) = L_{n,p}f(\|x\|^2)$, where $L_{n,p}$ is the Jacobi operator described above.

For the Ornstein-Uhlenbeck operator, we may scale the operator $L_{n,n}$ by a factor $\sqrt{n}$, that is let it act on functions of the form $f(x/\sqrt{n})$, where the function f is compactly supported in $\mathbb{R}$. Then, when n goes to infinity, we get the Ornstein-Uhlenbeck operator. In this view, the Ornstein-Uhlenbeck operator may be seen as the limit of the radial part of the Laplace-Beltrami operator on $S_n(\sqrt{n})$. In the same way, the Gaussian measure is the limit of the projections of the measures of the uniform sphere on $S^n(\sqrt{n})$ onto the diameter. This is the celebrated Poincaré limit.

For the Laguerre operator with integer parameter, we may consider products of Ornstein-Uhlenbeck operators, that is

$$Lf(x) = \Delta f(x) - \sum_i x^i \frac{\partial f}{\partial x^i}$$

in $\mathbb{R}^n$. Then, we consider the radial part, that is the action of this operator on $F(x) = f(\|x\|^2)$: we get the Laguerre operator with parameter n. Of course, we may as well exchange projections and limits, that is project the sphere $S^m(\sqrt{m})$ onto an n-dimensional subspace (n being fixed), take radial parts, and then let m go to infinity. We see then that the Laguerre operator may be seen as a limit of operators $L_{n,m}$, scaled by the factor $\sqrt{m}$, when m goes to infinity. This limit is still valid when n is not an integer.

For these families of polynomials, it is even possible to describe all sequences (λ_k) which are the eigenvalues of generators of Markov semigroups. This had been done in [23, 24] for the case of the Jacobi polynomials, in [35] for the case of the Hermite polynomials and in [12] for the case of Laguerre polynomials (see [12] for details and more examples).

These values of (λ_k) are

• For the Jacobi polynomials :

$$\lambda_k = \theta k(k + \frac{m+n}{2} - 1) + \rho \int_{-1}^{1} \frac{P_k(1) - P_k(x)}{1-x} \nu(dx),$$

where θ and ρ are positive constants and ν is a probability measure on $[-1, 1]$.

• For the Hermite polynomials,

$$\lambda_k = \theta k + \rho \int_{-1}^{1} \frac{1-x^k}{1-x} \nu(dx),$$

where θ and ρ are positive constants and ν is a probability measure on $[-1, 1]$.

• For the Laguerre polynomials,

$$\lambda_k = \theta k + \rho \int_{0}^{1} \frac{1-x^k}{1-x} \nu(dx),$$

where θ and ρ are positive constants and ν is a probability measure on $[0, 1]$.

Let us give some short indications about the proof of this kind of results. They come easily from the same representation of Markov operators, i.e. the possible values of μ_k for which the operator K defined by $KP_k = \mu_k P_k$ is positivity preserving with $K1 = 1$.

The result for Jacobi polynomials then relies on a theorem of Gasper [23, 24], which asserts that, for the family (P_k) of Jacobi polynomials, the function

$$M(x, y, z) = \sum_n \frac{P_n(x)P_n(y)P_n(z)}{P_n(1)}$$

is positive (and of course that the series converges in $L^2(\mu^{\otimes 3})$).

Then, this kernel allows us to construct a convolution on probability measures by

$$\rho_1 * \rho_2(dz) = [\int M(x,y,z)\rho_1(dx)\rho_2(dy)]\, \mu(dz).$$

This convolution is commutative and has δ_1 as identity element. In the same way, we may define a convolution of two integrable functions by identifying the function f with the measure $f(x)\mu(dx)$.

From the very definition of M it is quite obvious that $P_n * P_m = \delta_{m,n}\dfrac{P_n}{P_n(1)}$. Then, if K is a MARKOV kernel, we have $K(f*g) = K(f)*g = f*K(g)$. Then, the representation of K is given by $K(f) = K(\delta_1)*f$. If $\nu(dx) = K(\delta_1)$, then we see from the definition of M that

$$\mu_k = \int \frac{P_n(x)}{P_n(1)}\,\nu(dx).$$

From this, the representation of the eigenvalues of generators of MARKOV semigroups may be deduced by standard arguments.

The case of HERMITE and LAGUERRE polynomials is a bit more complicated, since then the measure ν is formally $K(\delta_\infty)$, and the convolution structure is degenerated. But the proof follows essentially along the same lines, see [12].

1.7.4 The infinite dimensional Ornstein-Uhlenbeck operator.

Finally, let us present an infinite dimensional example, which illustrates the difficulties with using the distance function in infinite dimension. The space E is this time the space $C_0([0,1])$ of continuous functions f on the interval $[0,1]$ such that $f(0)=0$. The σ-algebra is the smallest one for which the coordinate functions $B_t(f) = f(t)$ are measurable, and the measure μ is such that $t \mapsto B_t(f)$ is a standard one-dimensional Brownian motion: it is called the standard Wiener measure. Let us chose an orthonormal basis of $L^2([0,1])$, say (f_n), such that $f_0 = 1$, and consider the applications $X_n(f) = \int_0^1 f_n(s)dB_s(f)$, where the integral denotes the stochastic integral with respect to the Brownian motion. Then, under the measure μ, these functionals are independent standard Gaussian variables. Let $\mathcal{A}$ be the algebra of all functions on E which are polynomials in a finite number of the functions $X_n(f)$, say $G(f) = F(X_0,\dots,X_n)$. On such a function, we define the operator LF as $LG(f) = K(X_0,\dots,X_n)$, where

$$K(X_0,\dots,X_n) = \sum_i \frac{\partial^2 F}{\partial^2 X_i} - \sum_i X_i \frac{\partial F}{\partial X_i}.$$

This defines the generator of a MARKOV diffusion semigroup, known as the infinite dimensional ORNSTEIN-UHLENBECK operator on E, which has μ as reversible measure.

Now, a precise analysis of this operator (see [28]) shows that $d(f,g)$ is finite if and only if the function $f-g$ belongs to the Cameron-Martin space of functions absolutely continuous and having a square integrable derivative in $L^2([0,1])$. This Cameron-Martin space has μ-measure 0. Therefore, for any $f \in E$, the set of points g such that $d(f,g)$ is finite has measure 0. But, for a proper definition of the distance to a set A, the set of points $g \in E$ such that $d(A,g)$ is finite is of measure 1 as soon as $\mu(A) > 0$. We therefore see that in this setting the distance function to a point is not a very useful tool, and has to be replaced by distances to sets.

2 Functional inequalities

There are many interesting inequalities which relate L^p norms of a function to L^q norms of it's gradient, where the gradient is understood as $\Gamma(f,f)^{1/2}$, the reference measure being the invariant measure μ. In this section, we shall concentrate on few of them, which all belong to the family

$$\varepsilon \|f\|_p^2 \leq A\|f\|_2^2 + B\int \Gamma(f,f)\, d\mu,$$

where ε is the sign of $p-2$.

In all what follows, we shall denote $\mathcal{E}(f,f)$ the quantity $\int \Gamma(f,f)\, d\mu$, which stands for the energy of the function f, and is called the DIRICHLET *form. Also, to simplify the notation, we shall denote $\langle f\rangle = \int f\, d\mu$, and $\langle f,g\rangle = \int fg\, d\mu$.*

For example, on the unit sphere S_n $(n > 2)$, one has for the normalized RIEMANN measure

$$\frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \leq \frac{1}{n}\mathcal{E}(f,f),$$

for any $p \in [1, 2n/(n-2)]$. The constants in these inequalities are sharp, and the maximal value of p is also critical. The meaning of these inequalities is quite different when $p \in [1,2)$ or when $p \geq 2$. Here, we shall mainly concentrate on the three cases $p = 1$ (spectral gap inequalities), $p = 2$, which is a limiting case (logarithmic SOBOLEV inequalities), and $p > 2$ (SOBOLEV inequalities).

In general, for some choice of the coefficients in the inequality, one may deduce from the inequality that a function which satisfies $\mathcal{E}(f,f) = 0$ is constant. In this case, we say that the inequality is tight, and tight inequalities will in general lead to ergodic properties, and more precisely to the control of the convergence to equilibrium when t tends to infinity.

There is a lot to be said about other families of inequalities : those concerning the L^1 norm of the gradient are isoperimetric inequalities or Scheeger inequalities, and are in general stronger than the corresponding ones concerning the L^2 norm of the gradient. But there are also weaker inequalities such as the uniform positivity improving inequalities, which may be useful in different contexts (see [1], for example).

In all what follows, the inequalities do not refer explicitly to the semigroup P_t, but to the operator Γ and to the measure μ. We are then therefore concerned by the structure (E, Γ, μ), where E is the space, Γ is the carré du champ operator, and μ is the invariant measure. As we already saw, if the operator L is symmetric, it is entirely characterized by these data.

If the operator L is not symmetric, the symmetric operator described by Γ and μ is $(L+L^*)/2$, where the * denotes the adjoint in $L^2(\mu)$. From this point of view, the inequalities described below are in fact inequalities concerning the symmetric operator $(L+L^*)/2$. This is always the generator of a MARKOV semigroup. Indeed, the adjoint semigroup P_t^* is MARKOV, since if f and g are positive functions,

$$\langle P_t^*(f), g\rangle = \langle f, P_t g\rangle \geq 0,$$

which shows that P_t^* is positivity preserving, and

$$\langle P_t^*(1), g\rangle = \langle P_t g\rangle = \langle g\rangle,$$

so that $P_t^*(1) = 1$. Therefore, for any convex function ϕ, one has $(L+L^*)(\phi(f)) \geq \phi'(f)(L+L^*)(f)$, which shows that $(L+L^*)/2$ is the generator of a MARKOV semigroup.

It might happen that the carré du champ of $(L+L^*)/2$ is not Γ. This is never the case when the operator L is a diffusion operator. But in any case, the DIRICHLET form $\mathcal{E}(f,f)$ associated to $(L+L^*)/2$ is the same as the one associated to L.

It may therefore seem curious that some of the results which relate the inequalities and the behaviour of the semigroup still hold in the non symmetric case. This just means that those results are valid uniformly for the full class of operators having the same measure μ and the same carré du champ.

2.1 Spectral Gap.

In this section, we are concerned with the case where μ is a probability measure. Then, we say that L satisfy the spectral gap inequality with a constant C if and only if, for any function $f \in \mathcal{A}$, one has

$$\int f^2\, d\mu \leq (\int f\, d\mu)^2 + C\mathcal{E}(f,f). \tag{14}$$

The best constant C for which the inequality holds is called the spectral gap constant.

We may rewrite the inequality (14) in the form

$$\sigma^2(f) \leq C\mathcal{E}(f,f),$$

where $\sigma^2(f)$ denotes the variance of the function f with respect to the measure μ.

When the operator L is symmetric, we will prove below that the spectral gap inequality is nothing else than the fact that the spectrum of $-L$ is included in $\{0\}\cup[1/C,\infty)$. Therefore, the spectrum of $-L$ has a gap between 0 and $1/C$.

In fact, when L is symmetric, it is straightforward to check that, if $\lambda \neq 0$ is an eigenvalue of $-L$, with eigenvector f in $L^2(\mu)$, then one has $\int f\, d\mu = 0$ (since f is orthogonal to the constant function 1 which is the eigenvector associated to the eigenvalue 0), and then

$$\int f^2\, d\mu = \frac{1}{\lambda}\int -Lf\, f\, d\mu = \frac{1}{\lambda}\mathcal{E}(f,f),$$

so that $\frac{1}{\lambda} \leq C$.

On the other hand, suppose that $-L$ is symmetric and the the spectrum of $-L$ lie in $\{0\}\cup[1/\lambda,\infty)$. By the spectral theorem [41], if $\int f\, d\mu = 0$, then the spectral decomposition of $-L$ is

$$-Lf = \int_\lambda^\infty t dE_t(f),$$

and we have

$$\mathcal{E}(f,f) = \int_\lambda^\infty t dE_t(f,f) \geq \lambda \int_\lambda^\infty dE_t(f,f) = \lambda\int f^2\, d\mu.$$

This gives the spectral gap inequality.

In the non symmetric case, we have the following translation on the semigroup of the spectral gap inequality.

Proposition 2.1 *A semigroup (P_t) satisfies the spectral gap inequality with constant C if and only if, for any function f in $L^2(\mu)$, one has*

$$\sigma^2(P_t f) \leq e^{-2t/C}\sigma^2(f). \tag{15}$$

Proof. — To go from equation (15) to the spectral gap inequality (14), take a function f such that $\int f d\mu = 0$, and observe that, when $t \to 0$, we have

$$\int (P_t f)^2 d\mu = \int f^2 d\mu + 2t \int f L f d\mu + o(t).$$

If we compare the two members of the inequality (15) when $t \to 0$, we get the spectral gap inequality (14) with the constant C.

For the converse, it is enough to prove it for a function f in the domain which satisfies $\int f d\mu = 0$. We have $\int P_t f \, d\mu = \int f \, d\mu = 0$. If we set $\phi(t) = \int (P_t f)^2 \, d\mu$, we get

$$\phi'(t) = 2 \int L(P_t f) P_t f \, d\mu = -2\mathcal{E}(P_t f, P_t f).$$

Therefore, the spectral gap inequality gives

$$\phi'(t) \leq -\frac{2}{C}\phi(t),$$

from which we deduce that

$$\phi(t) \leq e^{-2t/C}\phi(0).$$

∎

Thus the spectral gap inequality implies an exponential rate at which the function $P_t f$ converges to $\int f \, d\mu$ in the L^2 norm as $t \to \infty$. In the symmetric case, this is exactly equivalent to the fact that the spectrum lies in $\{0\} \cup [1/C, \infty)$.

The spectral gap property has two important features :

1. It is stable under tensorization : if the spectral gap inequality holds on (E_1, Γ_1, μ_1) with a constant C_1 and on (E_2, Γ_2, μ_2) with a constant C_2, then is is true for $(E_1 \times E_2, \Gamma_1 + \Gamma_2, \mu_1 \otimes \mu_2)$ with the constant $C = \max(C_1, C_2)$. This property is obvious if we remark that, for a function $f(x,y)$,

$$\begin{aligned} \int\int f^2(x,y)\,\mu_1(dx)\mu_2(dy) - (\int\int f(x,y)\,\mu_1(dx)\mu_2(dy))^2 &= \\ \int (\int f^2(x,y)\,\mu_2(dy) - (\int f(x,y)\,\mu_2(dy))^2)\,\mu_1(dx) &+ \\ \int h^2(x)\,\mu_1(dx) - (\int h(x)\,\mu_1(dx))^2, \end{aligned}$$

   where $h(x) = \int f(x,y)\,\mu_2(dy)$.

2. The spectral gap inequality is stable under bounded perturbations. If we replace Γ by Γ_1 with $\Gamma_1 \leq a\Gamma$, then, the spectral gap inequality for (Γ, μ) with a constant C implies the spectral gap inequality for (Γ_1, μ) with the constant aC.

   Also, if we replace the measure μ by a measure μ_1 such that $\frac{1}{a}\mu \leq \mu_1 \leq a\mu$, then the spectral gap inequality for (Γ, μ) with a constant C implies the spectral gap inequality with the constant $a^3 C$ for (Γ, μ_1). This comes from the fact that we may write the variance of a function f as

$$\frac{1}{2}\int\int (f(x) - f(y))^2 \, \mu(dx)\mu(dy).$$

   (Recall that μ is a probability measure in this Section.)

A typical example of a measure satisfying the spectral gap inequality on $\mathbb{R}$ with $\Gamma(f, f) = f'^2$ is the measure $\mu(dx) = e^{-|x|}dx/2$. Let us show this property, with an argument which may be easily extended to more general settings. Here, the operator L associated to this measure is $Lf = f'' - \varepsilon(x)f'$, where $\varepsilon(x)$ denotes the sign of x. (There is a discontinuity in 0 but it does not really matter : from what we saw about the perturbation of the measure, we may replace the function $|x|$ by a small perturbation and go to the limit, if we want.) From the expression of L we see that, if $u(x) = |x|$, then $L(u) = 2\delta_0 - 1$, while $\Gamma(u, u) = 1$.

Then, we may write for any compactly supported smooth function f,

$$\int (f - f(0))^2 \, d\mu = -\int Lu(f - f(0))^2 \, d\mu = 2\int (f - f(0))\Gamma(u, f) \, d\mu.$$

Then, we use Schwarz' inequality to say that

$$|\Gamma(f, u)| \leq \sqrt{\Gamma(f, f)}\sqrt{\Gamma(u, u)} = \sqrt{\Gamma(f, f)},$$

then Schwarz' inequality again to get

$$\int (f - f(0))^2 \, d\mu \leq 2\sqrt{\int (f - f(0))^2 \, d\mu}\sqrt{\mathcal{E}(f, f)}.$$

From this, we get

$$\int (f - f(0))^2 \, d\mu \leq 4\mathcal{E}(f, f),$$

and since

$$\sigma^2(f) \leq \int (f - f(0))^2 \, d\mu,$$

we get the spectral gap inequality (14) with a constant 4. It is a bit surprising that such a crude argument gives the optimal constant C, but it is indeed the case as we shall see later on.

There is a general criterion, due to Muckenhoupt which characterizes the probability measures μ on $\mathbb{R}$ which satisfy the spectral gap inequality, with $\Gamma(f, f) = f'^2$. Namely, we have

Proposition 2.2 *Let μ a probability measure on $\mathbb{R}$. Then, if the measure μ satisfies the spectral gap inequality with respect to the carré du champ operator $\Gamma(f,f) = f'^2$, then μ must have a density ρ with respect to the* LEBESGUE *measure. Let m be a median of μ, i.e. any point such that $\mu([m,\infty)) = \mu((-\infty,m]) = 1/2$. Consider the quantities*

$$B_+ = \sup_{x>m} \mu([x,\infty)) \int_m^x \frac{1}{\rho(t)} dt$$

and

$$B_- = \sup_{x<m} \mu() - \infty, x]) \int_x^m \frac{1}{\rho(t)} dt,$$

then, the measure μ has a spectral gap if and only if the quantity $B = \max(B_+, B_-)$ is finite, and the best spectral gap constant C satisfies

$$B \leq C \leq 4B.$$

The proof of this result may be found in the book [3], for example.

If a measure μ satisfies the spectral gap inequality, then every integrable LIPSCHITZ function is exponentially integrable. This had been investigated first in [25], and more recently in a wider setting in [2]. We follow the arguments presented in [29] in the simpler context of diffusion semigroups.

Proposition 2.3 *Suppose that L is a diffusion operator which satisfies the spectral gap inequality with a constant C. Then, if f satisfies $\Gamma(f,f) \leq 1$ and $\int f\, d\mu < \infty$, we have*

$$\int e^{\lambda f}\, d\mu < \infty,$$

for any $\lambda < \sqrt{4/C}$.

Proof. — In fact, we shall prove that for these values of λ one has

$$\langle e^{\lambda f} \rangle \leq e^{\lambda \langle f \rangle} \prod_{k=0}^{\infty} (1 - C\frac{\lambda^2}{4^k})^{-2^k}.$$

To prove this, we may as well replace f by $f_n = (-n) \vee f \wedge n$, since $\Gamma(f_n, f_n) \leq 1$, and then go to the limit, using FATOU's Lemma. If we do not want to work with non-smooth functions, we may as well replace the functions $-n \vee x \wedge n$ by sequence of smooth bounded LIPSCHITZ functions which converge to x when $n \to \infty$.

Therefore, we may restrict ourselves to the case where f is bounded. Then, we apply the spectral gap inequality to $g = e^{\lambda f/2}$, and since

$$\mathcal{E}(g,g) = \frac{\lambda^2}{4} \int e^{\lambda f} \Gamma(f,f)\, d\mu \leq \frac{\lambda^2}{4} \int e^{\lambda f}\, d\mu,$$

one gets, setting $\phi(\lambda) = \int e^{\lambda f}\, d\mu$,

$$\phi(\lambda)(1 - C\lambda^2/4) \leq \phi^2(\lambda/2).$$

If $(1 - C\lambda^2/4) > 0$, then we have

$$\phi(\lambda) \leq \phi(\lambda/2)^2 (1 - C\lambda^2/4)^{-1},$$

and it remains to iterate the procedure replacing λ by $\lambda/2$ to get the result.

Notice that in the case of the measure $e^{-|x|}\,dx$, one has $C = 4$ and we get as critical exponent $\lambda = 1$: this shows at the same time that the previous result is optimal, and that the constant 4 in this example is optimal too. We refer to [29] for the non diffusion case. ∎

2.2 Logarithmic Sobolev Inequalities

We now turn to the special case where $p = 2$. In this chapter, we restrict ourselves again to the case when μ is a probability measure. We define the entropy of a positive function f to be

$$\mathrm{Ent}(f) = \int (f \log f)\, d\mu - \int f\, d\mu \log(\int f\, d\mu).$$

Since μ is a probability measure, then this is always a positive quantity by Jensen's inequality. Observe also that, since $x \log x$ is strictly convex, then only the constant functions have 0 entropy. Notice also that $\mathrm{Ent}(cf) = c\mathrm{Ent}(f)$.

A logarithmic Sobolev inequality has the form

$$\forall f \in \mathcal{A},\ \mathrm{Ent}(f^2) \leq A\langle f^2 \rangle + C\mathcal{E}(f, f). \tag{16}$$

Although this inequalities proved to be very useful in the non-diffusion case (see [34], e.g.), we shall restrict ourselves here to the diffusion setting.

These inequalities were introduced by L. Gross in [26] to prove the hypercontractivity result of Nelson (see [32]) for the Ornstein-Uhlenbeck semigroup. It turned out that this inequality is valid in a wide range of settings, in particular in many infinite dimensional situations.

From what we saw, when the measure μ is a probability measure, then the inequality is tight as soon as $A = 0$ in (16). (Remind that tightness in an inequality just means that $\Gamma(f, f) = 0$ implies that f is constant.)

A first important remark is to observe that the tight logarithmic Sobolev inequality with constant C implies the spectral gap inequality with constant $C/2$. To see this, it suffices to apply the inequality (16) to $1 + \varepsilon f$ and let ε go to 0.

Conversely, if a non tight logarithmic Sobolev inequality holds together with the spectral gap inequality, then the tight logarithmic Sobolev inequality holds. To see this, we may use an inequality of Rothaus (see [33]), which asserts that, for any square integrable function f, if we set $\hat{f} = f - \langle f \rangle$, then

$$\mathrm{Ent}(f^2) \leq 2\sigma^2(f) + \mathrm{Ent}(\hat{f}^2).$$

Then we apply the logarithmic Sobolev inequality to $\hat{f}$ instead of f, and then apply the spectral gap inequality.

Therefore, if there exists a logarithmic Sobolev inequality, then tightness is equivalent to the spectral gap inequality.

The tight logarithmic Sobolev inequality leads to exponential decay of the entropy for large t. We have

Proposition 2.4 *The tight logarithmic* Sobolev *inequality holds with a constant C if and only if, for any integrable positive function f,*

$$\mathrm{Ent}(P_t f) \leq e^{-4t/C}\mathrm{Ent}(f). \tag{17}$$

Proof. — The argument is completly similar to the exponential decay of the variance under spectral gap. If we have the inequality (17), then we may divide the difference by t and take the limit when t goes to 0 : we get the inequality (16) with $A = 0$.

Conversely, the time derivative of $H(t) = \langle \phi(P_t f) \rangle$ is $-\langle \phi''(P_t f)\Gamma(P_t f, P_t f)\rangle$, and, with the function $\phi(x) = x \log x$, the inequality (16) with $A = 0$ gives $H'(t) \leq -4H(t)/C$, from which we get the exponential decay.

∎

The main important fact about the logarithmic SOBOLEV inequality is GROSS' theorem [26], which relies the inequality to the hypercontractivity property of the semigroup. Recall that we restrict ourselves to the diffusion setting here (this is not compulsory), but we do not require the measure μ to be reversible.

The first thing to do is to write a slightly modified version of the logarithmic SOBOLEV inequality (16): for any smooth function ϕ, we have the identity

$$\langle \phi(f) Lf \rangle = -\langle \phi'(f)\Gamma(f,f)\rangle.$$

It is noticeable that this identity does not in fact require symmetry, but only the invariance property, since it says that $\langle L(\Phi(f))\rangle = 0$, where $\Phi' = \phi$. We also have

$$\Gamma(\phi(f), \phi(f)) = \phi'^2(f)\Gamma(f,f).$$

Using this, if we change f into $f^{p/2}$ in the logarithmic SOBOLEV inequality (16), we get the following

$$\mathrm{Ent}(f^p) \leq A\langle f^p \rangle + C\frac{p^2}{4}\int f^{p-2}\Gamma(f,f)\, d\mu = A\langle f^p\rangle - C\frac{p^2}{4(p-1)}\int f^{p-1}Lf\, d\mu. \quad (18)$$

This form of the logarithmic SOBOLEV inequality is the key point to get the following result:

Theorem 2.5 *Let A and C be two positive constants, $p \in (1, \infty)$ and let the functions $q(t)$ and $m(t)$ be defined by*

$$\frac{q(t)-1}{p-1} = \exp(\frac{4t}{C}); \qquad m(t) = \frac{A}{16}(\frac{1}{p} - \frac{1}{q(t)}).$$

Then, the following statements are equivalent :

1. *The logarithmic* SOBOLEV *inequality (16) is satisfied with constants A and C;*

2. *For any $t > 0$, any $f \in L^p(\mu)$,*

$$\|P_t f\|_{q(t)} \leq \exp(m(t))\|f\|_p.$$

In other words, the logarithmic SOBOLEV inequality is equivalent to the fact that the semigroup maps L^p into $L^{q(t)}$, with a norm which is controlled through the constant A. Observe that, when $A = 0$ (case of tight logarithmic SOBOLEV inequalities), then the norm is 1.

We shall not give any detail of the proof here (the reader may consult [3] or [5], for example). Let us just mention that the proof boils down to check that the logarithmic SOBOLEV inequality under the form given by (18) says that the derivative of

$e^{-m(t)}\|P_t f\|_{q(t)}$ is negative. The logarithmic SOBOLEV inequality is just the derivative in $t = 0$ of the hypercontractivity property.

In the symmetric case, it had been observed in [37] that it is enough to know that, for some t, and some $q > p$, the operator P_t is bounded from L^p into L^q to obtain the logarithmic SOBOLEV inequality. Moreover, if the norm in this case is 1, one gets the tight logarithmic SOBOLEV inequality. This result requires the symmetry property because it relies on the complex interpolation theorem which is only valid in this case (see [5] for more details).

Recently, F.Y. WANG even proved a stronger result : namely as soon as P_t is bounded for some t from L^2 into L^4 with norm strictly less than 2, then the spectral gap inequality holds, and therefore the tight logarithmic SOBOLEV inequality holds too (see [40]).

The tight logarithmic SOBOLEV inequality shares the same properties as the spectral gap inequality: stability under tensorization, under bounded perturbations of the measure, and under bounded perturbations of the square field operator.

Stability under tensorization is quite obvious under the hypercontractivity form of theorem 2.5, and the stability by bounded perturbation of the measure is slightly more tricky (see [3]).

Here, the typical measure on $\mathbb{R}$ which satisfies the logarithmic SOBOLEV inequality with $\Gamma(f,f) = f'^2$ is the Gaussian measure $e^{-x^2/2}\,dx/\sqrt{2\pi}$. In this case, the logarithmic SOBOLEV inequality is tight and the optimal constant C is equal to 2. There is no straightforward argument to see that, but we shall see a proof in Section 3, using curvature arguments. Observe that the constant is optimal since it implies the spectral gap inequality (14) with constant $C = 1$, and the spectrum of the ORNSTEIN-UHLENBECK operator, which is the symmetric operator associated to (Γ, μ) in this case, is $-\mathbb{N}$, as wee saw in Section 1.7.3: the spectrum is discrete and the first non zero eigenvalue of $-L$ is 1.

There is a general result which characterizes those measures on $\mathbb{R}$ which satisfy the tight logarithmic SOBOLEV inequality when $\Gamma(f,f) = f'^2$, similar to MUCKENHOUPT's result of Proposition 2.2, and which is due to BOBKOV and GÖTZE ([18]). Namely

Proposition 2.6 *Let μ be a probability measure on $\mathbb{R}$. Then, if the measure μ satisfies the tight logarithmic* SOBOLEV *inequality with respect to the carré du champ operator $\Gamma(f,f) = f'^2$, then μ must have a density ρ with respect to the* LEBESGUE *measure. Let m be a median of μ. Consider the quantities*

$$D_+ = \sup_{x>m} \mu([x,\infty)) \log(\frac{1}{\mu([x,\infty))}) \int_m^x \frac{1}{\rho(t)} dt,$$

and

$$D_- = \sup_{x<m} \mu((-\infty,x]) \log(\frac{1}{\mu((-\infty,x])}) \int_x^m \frac{1}{\rho(t)} dt.$$

Then, the measure μ satisfies the tight logarithmic SOBOLEV *inequality if and only if the quantity $D = \max(D_+, D_-)$ is finite, and the best constant C satisfies*

$$D/150 \leq C \leq 2880D.$$

We refer to [3] for the proof of this result.

In the case of tight logarithmic Sobolev inequalities, we also have special integrability properties for Lipschitz functions. The argument is due to Herbst :

Proposition 2.7 *If a measure μ satisfies the tight logarithmic* Sobolev *inequality, then every integrable* Lipschitz *function is exponentially square integrable. If the logarithmic* Sobolev *constant is C, then every* Lipschitz *integrable function with $\Gamma(f,f) \leq 1$ satisfies*

$$\int e^{\alpha f^2}\, d\mu < \infty,$$

for any $\alpha < 1/C$.

Moreover, we have the inequality

$$\langle e^{\alpha f^2} \rangle \leq \exp(\frac{\alpha}{1-C\alpha}\langle f\rangle^2)/\sqrt{1-C\alpha}.$$

We have of course a similar result with $\Gamma(f,f) \leq c^2$ changing f into f/c.

Proof. — We follow the proof of this theorem in [3]. As before, we may restrict ourselves to bounded Lipschitz functions. Applying the logarithmic Sobolev inequality to $\exp(\lambda f/2)$, and setting $H(\lambda) = \langle \exp(\lambda f)\rangle$, we observe that $\mathrm{Ent}(\exp(\lambda f)) = \lambda H'(\lambda)$. Then, the logarithmic Sobolev inequality gives a differential equation

$$\lambda H'(\lambda) \leq H \log H + C\lambda^2/4.$$

This is easily integrated between 0 and λ obtaining

$$H(\lambda) \leq \exp(\lambda\langle f\rangle + C\lambda^2/4).$$

Then, we integrate this inequality, which is valid for every λ, with respect to the measure $\exp(-t^2\lambda^2/2)d\lambda$ on $\mathbb{R}$. Since

$$\int \exp(tx - t^2/2)\, dt = \exp(x^2/2),$$

we get the result after a simple change of notations. ∎

One should notice that this result is optimal in the case of the Gaussian measure, where for the function x, the maximum value for α is $1/2$, while $C = 2$ (which gives a second argument why the constant 2 is optimal in the Gaussian case).

2.3 Sobolev inequalities.

The Sobolev inequalities we are interested in here take the form

$$\forall f \in \mathcal{A},\ \|f\|_p^2 \leq A\|f\|_2^2 + B\mathcal{E}(f,f), \tag{19}$$

where $A \in \mathbb{R}$, $B \geq 0$, and $p > 2$. In general, we shall set $p = 2n/(n-2)$, and call n the dimension in the Sobolev inequality.

The reason of this definition comes from the main examples where this inequality holds : on a compact n-dimensional Riemannian manifold ($n \geq 3$), with the Riemann measure, such an inequality always holds, $p = 2n/(n-2)$ being the best exponent.

The basic examples where the best constants in SOBOLEV inequalities are known are those of model spaces in Riemannian geometry, that is spheres, Euclidean spaces and hyperbolic spaces.

More precisely, let us denote by Γ_S, Γ_E and Γ_H the carrés du champ of these models in dimension n. Let ω_n be the surface measure of the unit sphere S_n in $\mathbb{R}^{n+1}$. For these three models, we chose as reference measures the Riemann measures divided by the constant ω_n, say μ_S, μ_E and μ_H, so that μ_S is a probability measure. Then, we have

1. For spheres $\|f\|^2_{2n/(n-2)} \leq \|f\|_2^2 + \frac{4}{n(n-2)} \int \Gamma_S(f,f) d\mu_S$;
2. For Euclidean spaces, $\|f\|^2_{2n/(n-2)} \leq \frac{4}{n(n-2)} \int \Gamma_E(f,f) d\mu_E$;
3. For hyperbolic spaces $\|f\|^2_{2n/(n-2)} \leq -\|f\|_2^2 + \frac{4}{n(n-2)} \int \Gamma_H(f,f) d\mu_H$.

We shall see later (in Section 4) how one may go from one of these inequalities to the others through conformal transformations. Also, we shall give in the next section a hint on how to obtain the optimal SOBOLEV inequality on the sphere. But we may observe already that, in the hyperbolic case, the SOBOLEV inequality implies that $\|f\|_2^2 \leq \frac{4}{n(n-2)} \int \Gamma_H(f,f) d\mu_H$. This is not an optimal result, since one may prove that the optimal inequality in this context is

$$\|f\|_2^2 \leq \frac{4}{(n-1)^2} \int \Gamma_H(f,f) d\mu_H,$$

which in fact says that the spectrum of the generator $-\Delta_H$ (the hyperbolic LAPLACE-BELTRAMI operator) lies in $[\frac{4}{(n-1)^2}, \infty)$: here, the measure is infinite, the constant function 1 is no longer in L^2, and the point 0 is no longer in the spectrum.

On the real line, the SOBOLEV inequality acting on the radial function on the sphere gives the SOBOLEV inequality for the symmetric JACOBI operator $L_{n,n}$ introduced in Section 1.7.3, with the same constant. The inequality remains true for any $n > 2$, even if it is not an integer.

To come back to the general setting, in the case where μ is a probability measure, the inequality is tight when $A = 1$. As for the logarithmic SOBOLEV inequality, when a SOBOLEV inequality holds with a probability measure μ, then tightness is equivalent to the spectral gap inequality (14). More precisely, if the tight SOBOLEV inequality holds with the constant B, then the spectral gap inequality (14) holds with constant $C = B/(p-1)$. This is obtained by taking $f = 1 + \varepsilon g$, and letting ε go to 0.

Conversely, we may prove that, for a probability measure μ, we have for any $p > 2$

$$\|f\|_p^2 \leq \|f\|_2^2 + (p-1)\|\hat{f}\|_p^2,$$

where $\hat{f} = f - \int f \, d\mu$ (see [5]). Therefore, when a non tight SOBOLEV inequality holds, we may first use this inequality and then apply the SOBOLEV inequality to $\hat{f}$ to get a tight SOBOLEV inequality.

There are many different forms under which one may encounter a SOBOLEV inequality. For example, the family of GAGLIARDO-NIRENBERG inequalities which take the form

$$\|f\|_r \leq \|f\|_s^{1-\theta}(A\|f\|_2^2 + B\mathcal{E}(f,f))^{\theta/2},$$

where $\theta \in [0,1]$, $r \geq 1$, $s \geq 1$ and $\frac{1}{r} = \frac{\theta(n-2)}{2n} + \frac{1-\theta}{s}$.

These inequalities are easily deduced from the SOBOLEV inequality by HOLDER's inequality, with the same constants A and B. Conversely, to go from any of these inequalities to the SOBOLEV inequality (with possibly different constants), we first restrict ourselves to positive functions, then apply the last inequality to $f_k = 2^k \vee f \wedge 2^{k+1}$ and finally sum over all possibles values of $k \in \mathbb{Z}$. The basic ingredient here is that $\mathcal{E}(\phi(f), \phi(f)) \leq \mathcal{E}(f, f)$ whenever ϕ is a contraction (contractions leave the domain of DIRICHLET forms invariant), and that

$$\sum_{k \in \mathbb{Z}} \mathcal{E}(f_k, f_k) \leq \mathcal{E}(f, f).$$

(See [9] for more details, and a wider class of equivalent inequalities.)

Of particular interest is the case $r = 2$, $s = 1$, which the NASH' inequality, and the limiting case $r = s = 2$, when the inequality takes the form

$$\|f\|_2 = 1 \implies \mathrm{Ent}(f^2) \leq \frac{n}{2} \log[A + B\mathcal{E}(f, f)], \tag{20}$$

which looks like the logarithmic SOBOLEV inequality but in fact shares the properties of SOBOLEV inequalities : we call it the logarithmic entropy-energy inequality.

This last inequality belongs to a wider class of entropy-energy inequalities, which take the form

$$\|f\|_2 = 1 \implies \mathrm{Ent}(f^2) \leq \Phi(\mathcal{E}(f, f)), \tag{21}$$

where Φ is an increasing concave function. Such an inequality is tight when $\Phi(0) = 0$. If the derivative of Φ in 0 is finite, then it implies the logarithmic SOBOLEV inequality. Therefore, a tight SOBOLEV inequality implies the tight logarithmic SOBOLEV inequality.

The main feature of the SOBOLEV inequality is that it implies a strong bound on the semigroup P_t. In fact, one has

Theorem 2.8 *If the SOBOLEV inequality holds with dimension n, then P_t maps L^1 into L^∞ with a norm bounded from above by $Kt^{-n/2}$ for $t \in (0,1)$.*

Conversely, if the semigroup is symmetric and bounded from L^1 into L^∞ with a norm bounded above by $Kt^{-n/2}$ for $t \in (0,1)$, then it satisfies a SOBOLEV inequality with dimension n.

The direct part of this result had been obtained by DAVIES ([20]), using families of logarithmic SOBOLEV inequalities which in fact are equivalent to the logarithmic entropy-energy (20) inequality stated above.

The converse is due to VAROPOULOS ([38]).

Proof. — We shall not enter in the details of the proof here. The equivalence is much easier to obtain with NASH inequalities (but not with optimal constants). Using the fact that, for a positive function f, $\|P_t f\|_1$ is constant, the NASH inequality gives a differential inequality on $\phi(t) = \|P_t f\|_2^2$ which has the form, when f is non negative and $\int f d\mu = 1$,

$$\phi(t) \leq [A\phi - 2B\phi']^\theta.$$

With $\theta = \frac{n}{n+2}$, and setting $\phi = \psi^{-n/2}$, we get

$$nB\psi' + A\psi \geq 1,$$

from which we get easily $\phi(t) \leq Ct^{-n/2}$ if $t \in (0,1)$. (This argument gives even a far more precise result.)

This shows that P_t maps L^1 into L^2 with a norm bounded by $Ct^{-n/4}$.

By duality, the same is true between L^2 and L^∞, and the final result comes from the fact that $P_t = P_{t/2} \circ P_{t/2}$.

The converse may be shown quite easily. If P_t maps L^1 into L^2 with a norm bounded from above by $Ct^{n/2}$ for $t \in (0,1)$, then one uses the fact that $\mathcal{E}(P_t f, P_t f)$ is decreasing in t (this relies on the spectral decomposition and requires the symmetry). Then, one writes, using the same notations as before,

$$\phi(t) = \phi(0) - 2\int_0^t \mathcal{E}(P_s f, P_s f) ds \geq \phi(0) - 2t\mathcal{E}(f,f).$$

Then, if $\phi(t) \leq Ct^{-n/2}$, one gets a majorization of $\phi(0)$ by $\mathcal{E}(f,f)$, that we may optimize in $t \in (0,1)$ to get the Nash inequality. ∎

This theorem gives much stronger results than the hypercontractivity estimate. Indeed, it implies that the semigroup $P_t(x,dy)$ has a density $p_t(x,y)$ with respect to the invariant measure $\mu(dy)$ which is bounded from above by $ct^{-n/2}$. In the finite measure case, the operator P_t is then Hilbert-Schmidt and the generator L has a discrete spectrum. One may also obtain precise bounds on the trace of P_t and on the norms of the eigenfunctions.

The use of energy-entropy inequalities (21) gives more precise information. Indeed, we have

Theorem 2.9 *If the energy-entropy inequality (21) holds with a concave C^1 function Φ, then if we set $\Psi(x) = \Phi(x) - x\Phi'(x)$, for any $\lambda > 0$, and any $q > p > 1$,*

$$\|P_{t_{p,q}}\|_{p\to q} \leq e^{m_{p,q}},$$

where

$$t_{p,q} = t_{p,q}(\lambda) = \frac{1}{2}\int_p^q \Phi'(\frac{\lambda s^2}{s-1})\frac{ds}{4(s-1)},$$

and

$$m_{p,q} = m_{p,q}(\lambda) = \frac{1}{2}\int_p^q \Psi(\frac{\lambda s^2}{s-1})\frac{ds}{s^2}.$$

The proof of this result follows the same lines that the proof of Gross' theorem, the trick being here to transform the energy-entropy inequality (21) into a family of logarithmic Sobolev inequalities depending on a parameter, and to adjust in an optimal way this parameter with the variable t. This is Davies' method. (See [5] or [3] for more details).

This method gives very precise estimates. Indeed, in $\mathbb{R}^n$, with the Lebesgue measure and the standard gradient, one may check, using the optimal logarithmic Sobolev inequality for the Gaussian measure that one has the energy-entropy inequality (21) with $\Phi(x) = \frac{n}{2}\log(2x/(n\pi e))$. If we explicit the above estimates with this function Φ, we get $p_t \leq (4\pi t)^{-n/2}$, which is the exact value of the upper bound on the density p_t of the heat kernel in $\mathbb{R}^n$. (See [8] for more details.)

This result about the relationship between SOBOLEV inequalities and bounds on the heat kernel shows immediately that the SOBOLEV inequality is not stable under tensorization : if it is true for E_1 with dimension n_1 and for E_2 with dimension n_2, then it is true for the product $E_1 \times E_2$ with dimension $n_2 + n_2$. It is not so easy to see that directly on the form of the inequality, but this is clear from the bounds on the heat kernel $p_t(x, dy)$. The best constants on the product one may obtain in this way is not known to us.

Moreover, under the SOBOLEV inequality, DAVIES obtained off diagonal bounds on the density $p_t(x, y)$ of the form $Kt^{-n/2}\exp(-\dfrac{d^2(x,y)}{4+\varepsilon})$, where d is the natural distance associated with the carré du champ Γ ([5]).

One may find in [8] examples of measures on the real line satisfying entropy-energy inequalities of various kind, with functions Φ behaving like x^α at infinity, for any $\alpha \in (0, 1)$.

Tight SOBOLEV inequalities, or tight entropy-energy inequalities lead to bounds on the diameter on one side, and to strong convergence to equilibrium on the other side.

For example, under the tight SOBOLEV inequality, one gets at infinity bounds of the form

$$-A\exp(-Ct) \leq \log(p_t(x, y)) \leq A\exp(-Ct),$$

where the constants A and C may be explicitely computed from the constant of the tight SOBOLEV inequality.

Also, if the tight SOBOLEV inequality holds, then the diameter is finite. This may be easily seen using the logarithmic entropy-energy (20) inequality and applying it to $\exp(\lambda f)$, where f is a LIPSCHITZ function.

But one may do better. Recall that, on the unit sphere S^n in $\mathbb{R}^{n+1}$, the tight SOBOLEV inequality holds with the constant $B = \frac{4}{n(n-2)}$. Then one can prove that, if the SOBOLEV inequality holds on some probability space with this constant, the diameter associated to its carré du champ is bounded from above by π, which is the diameter of the sphere. Of course, if the constant is different, we may scale the constant by a factor ρ and the diameter is scaled by a factor $1/\sqrt{\rho}$. But the proof of this comparison theorem is far much harder (see [11]).

3 Curvature-Dimension inequalities

We turn now to the study of the local structure of the generator, and more precisely to curvature-dimension inequalities which lead to the functional inequalities of the previous chapter. Once again, we shall restrict ourselves here to the study of diffusion operators, although there is a lot to be said about the non-diffusion case.

Let us introduce a new bilinear map $\mathcal{A} \times \mathcal{A} \mapsto \mathcal{A}$. In the same way that we defined the operator Γ from the standard product and the operator L, we define on $\mathcal{A}$

$$\Gamma_2(f, g) = \frac{1}{2}[L\Gamma(f, f) - \Gamma(f, Lg) - \Gamma(g, Lf)].$$

The reader may see that the construction of Γ_2 from Γ is similar to the construction of Γ from $\Gamma_0(f, g) = fg$. We could introduce in the same way operators Γ_k, $k \geq 3$.

They have not yet proved to be as useful as Γ_2, but one could see in [7] the use that can made of them. They are sometimes called the iterated carrés du champ.

When L is the LAPLACE-BELTRAMI operator of a Riemann structure, then one has

$$\Gamma_2(f,f) = |\nabla\nabla f|^2 + \mathrm{Ric}(\nabla f, \nabla f),$$

where $\nabla\nabla f$ is the (symmetric) second derivative of f computed in the Riemannian structure (it is a symmetric tensor), and $|\nabla\nabla f|^2$ denotes its HILBERT-SCHMIDT norm, that is the sum of squares of the coefficients of the matrix $\nabla\nabla f$ computed in an orthonormal basis, or the sum of squares of the eigenvalues of this symmetric matrix. The term (Ric) denotes the RICCI tensor of the Riemannian manifold : this is again a symmetric tensor, and $\mathrm{Ric}(\nabla f, \nabla f)$ denotes its action on the tangent vector (∇f).

Therefore, saying that the RICCI tensor of a manifold is bounded from below by ρ just means that $\mathrm{Ric}(\nabla f, \nabla f) \geq \rho|\nabla f|^2 = \rho\Gamma(f,f)$.

On the other hand, Δf is the trace of the tensor $\nabla\nabla f$, and we have

$$(\Delta f)^2 \leq n|\nabla\nabla f|^2.$$

This leads us to the following definition :

Definition 3.1 *We say that the operator L satisfies a $CD(\rho,n)$ inequality (curvature-dimension inequality with curvature ρ and dimension n) iff, for any function $f \in \mathcal{A}$, one has*

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{n}(Lf)^2.$$

Here, ρ is any real number and $n \in [1,\infty]$.

When $L = \Delta + \nabla h$, for the LAPLACE-BELTRAMI operator Δ of a given Riemannian structure g on an n-dimensional manifold, then

$$\Gamma_2(f,f) = |\nabla\nabla f|^2 + (\mathrm{Ric} - \nabla\nabla h)(\nabla f, \nabla f).$$

In particular, on the Euclidean space $\mathbb{R}^d$, with the usual gradient Γ_E, the structure $(\mathbb{R}^d, \Gamma_E, \mu)$ satisfies $CD(0,\infty)$ if and only if the measure is log-concave.

If the generator L is not symmetric, that is when it is written under the form $\Delta + X$, we just have to replace $\nabla\nabla h$ in the previous formula by $\nabla_S X$, which is the symmetrized covariant derivative of the vector field X. We shall denote the tensor $\mathrm{Ric} - \nabla_S X = \mathrm{Ric}(L)$, the RICCI tensor of the operator L.

Then, the $CD(\rho,m)$ inequality holds if and only if

$$m \geq n \quad \text{and} \quad (m-n)[\mathrm{Ric} - \nabla\nabla h - \rho g] \geq \nabla h \otimes \nabla h, \tag{22}$$

where inequality (22) has to be understood in the sense of symmetric tensors. In this view, the LAPLACE-BELTRAMI operators are among all generators of diffusion semigroups on a fixed manifold those which have the minimal dimension.

Notice that changing L into cL changes $CD(\rho,n)$ into $CD(c\rho,n)$. Therefore, we may always scale ρ by any given positive number, which amounts to rescale the time in the semigroup.

In this setting, the spherical Laplacian satisfies $CD(n-1,n)$, the Euclidean one satisfies $CD(0,n)$ and the hyperbolic one satisfies $CD(-(n-1),n)$.

In dimension one, let us write an operator on the real line as $Lf = f'' - a(x)f'$. Then, L satisfies $CD(\rho, n)$ if and only if

$$a' \geq \rho + \frac{a^2}{n-1}. \tag{23}$$

If we want to see what are the extremal cases, that is the case where there is equality in inequality (23), up to translation we find the following, which we call models of the $CD(\rho, n)$ inequality :

1. $CD((n-1), n)$: then, $a(x) = (n-1)\tan(x)$ on $(-\pi/2, \pi/2)$. When n is an integer, this is the radial part of the spherical Laplacian. Changing x into $\cos(x) = y$ gives the ultraspherical operator of the Section 1.7.3.

2. $CD(0, n)$: then, $a(x) = -(n-1)/x$. This is the radial part of the Euclidean Laplacian, when n is an integer. The associated operator is known as the Bessel operator.

3. : $CD(-(n-1), n)$. There are many solutions $a(x) = -(n-1)\text{cotanh}(x)$ on $(0, \infty)$, $a(x) = \pm(n-1)$ on $\mathbb{R}$, and $a(x) = -(n-1)\tanh(x)$ on $\mathbb{R}$. When n is an integer, the first one corresponds to the radial part of the hyperbolic Laplacian (i.e. the operator we get if the hyperbolic Laplacian acts on functions depending only on the hyperbolic distance to a point), the second one describes the action of the hyperbolic Laplacian on functions depending only on the distance to horocycles (i.e. spheres which are tangent to the unit sphere in the unit ball representation of the hyperbolic space), and the last one describes the action of the hyperbolic Laplacian on functions depending only on the hyperbolic distance to hyperbolic hyperplanes (i.e. hyperplanes passing through the origin in the unit ball representation).

4. $CD(1, \infty)$: then, $a(x) = x$, which represents the Ornstein-Uhlenbeck operator, which is the radial part of the spherical Laplacian on the sphere of radius $\sqrt{n-1}$ in dimension n when n goes to infinity. Notice that this sphere satisfies $CD(1, n)$, and that this $CD(1, n)$ inequality goes to the limit $CD(1, \infty)$.

As the previous examples show, even in dimension one, the $CD(\rho, n)$ captures the information of the space where the operator comes from, as far as Ricci curvature and dimension are concerned.

Let us summarize some of the results which are known about the relationship between the $CD(\rho, n)$ inequality and the functional inequalities of the previous sections.

Theorem 3.2 *1. If $CD(\rho, n)$ holds for some $\rho > 0$, then the invariant measure μ has to be finite. If it is reversible, then the tight* Sobolev *inequality holds. Moreover, for the invariant probability, the constant B in the* Sobolev *inequality is bounded above by $\frac{\rho}{n-1}\frac{4}{n(n-2)}$. Notice that this result is optimal for the spheres.*

2. If $CD(\rho, \infty)$ holds with $\rho > 0$, then the invariant measure has to be finite and, for the invariant probability, the tight logarithmic Sobolev *inequality holds with a constant C bounded above by $\frac{2}{\rho}$. Notice that this result is optimal for the Gaussian measure and the* Ornstein-Uhlenbeck *operator.*

3. *If $CD(0,\infty)$ holds, if the measure μ is a probability measure, and if the distance function is integrable, the logarithmic* Sobolev *inequality holds if and only if there exists $\alpha > 0$ such that $\int \exp(\alpha d^2(x,y))\mu(dy) < \infty$, for some (or equivalently for all) $x \in E$.*

4. *If $CD(-\rho,\infty)$ holds with some constant $\rho > 0$, then the logarithmic* Sobolev *inequality holds as soon as, for some $\alpha > \rho$, $\int \exp(\alpha d^2(x,y))\mu(dy) < \infty$, for some (or equivalently for all) $x \in E$.*

The proof of point (2) is quite easy, but the other results require a bit of technical work. Detailed proofs may be found in the book [3]. Recall that the one-dimensional models capture all the information about the $CD(\rho,n)$ inequalities. Therefore, the Ornstein-Uhlenbeck operator is a model for operators with positive curvature and infinite dimension, and it satisfies the logarithmic Sobolev inequality which is the typical infinite dimensional Sobolev inequality. The symmetric Jacobi operators are models for operators with positive curvature and finite dimension n, and satisfy the n-dimensional Sobolev inequality.

In fact, most of the results obtained under $CD(\rho,\infty)$ are easy to handle through the following, which relates the $CD(\rho,\infty)$ inequality to local functional inequalities, i.e. functional inequalities related to the heat kernel measures $P_t(x,dy)$ instead of the invariant measure μ.

Proposition 3.3 *Let P_t be a diffusion semigroup with generator L and ρ be any real number. Then, the following are equivalent :*

1. *$CD(\rho,\infty)$ holds.*

2. *$\forall f \in \mathcal{A}$, $\Gamma(P_tf,P_tf) \leq \exp(-2\rho t)P_t\Gamma(f,f)$.*

3. *$\forall f \in \mathcal{A}$, $\Gamma(P_tf,P_tf)^{1/2} \leq \exp(-\rho t)P_t(\Gamma(f,f)^{1/2})$.*

4. *$\forall f \in \mathcal{A}$, $P_tf^2 - (P_tf)^2 \leq \frac{1-\exp(-2\rho t)}{\rho}P_t\Gamma(f,f)$.*

5. *$\forall f \in \mathcal{A}$, $P_tf^2 - (P_tf)^2 \geq \frac{\exp(2\rho t)-1}{\rho}\Gamma(P_tf,P_tf)$.*

6. *For some $\alpha \in (1,2)$, $\forall f \in \mathcal{A}$,*
$$P_t(f^\alpha) - (P_tf)^\alpha \leq \alpha(\alpha-1)\frac{1-\exp(-2\rho t)}{2\rho}P_t(f^{\alpha-2}\Gamma(f,f)).$$

7. *For some $\alpha \in (1,2)$, $\forall f \in \mathcal{A}$,*
$$P_t(f^\alpha) - (P_tf)^\alpha \geq \alpha(\alpha-1)\frac{\exp(2\rho t)-1}{2\rho}(P_tf)^{\alpha-2}\Gamma(P_tf,P_tf).$$

8. *$\forall f \in \mathcal{A}$, $P_t(f\log f) - P_tf\log P_tf \leq \frac{1-\exp(-2\rho t)}{2\rho}P_t(\frac{\Gamma(f,f)}{f})$.*

9. *$\forall f \in \mathcal{A}$, $P_t(f\log f) - P_tf\log P_tf \geq \dfrac{\exp(2\rho t)-1}{2\rho}\dfrac{\Gamma(P_tf,P_tf)}{P_tf}$.*

Moreover, in (6) and (7), one may replace the function $x \mapsto x^\alpha$ by any convex function Φ such $\frac{1}{\Phi''}$ is concave. These inequalities are intermediate between the spectral gap inequality (4) and the logarithmic SOBOLEV *inequality(8).*

Of course, when $\rho = 0$, one has to replace $\dfrac{1-\exp(-2\rho t)}{\rho}$ by $2t$, and so on.

There are a lot of other different forms of the $CD(\rho,\infty)$ inequality (see [6], for example). The family of functional inequalities related to a generic function Φ has been extensively studied by D. CHAFAÏ in [19]. Let us concentrate on a few of them. The inequality (4) simply says that the family of measures $P_t(x, dy)$ satisfies the spectral gap inequality, uniformly in x, with a constant which depends only on t and goes to 0 when t goes to 0.

The inequality (8) tells the same thing with the logarithmic SOBOLEV inequality.

The reverse inequality (5) tells us for example that, if a function f is bounded, then $P_t f$ is LIPSCHITZ with the LIPSCHITZ norm behaving as $C/\sqrt{t}$ when $t \to 0$. This produces as many LIPSCHITZ functions as we want.

The reverse inequality (9) is more subtle. For simplicity, take the case $\rho = 0$. Then, as noticed by HINO in [27], when applied to an indicator function $\mathbb{1}_A$, it tells that $\sqrt{-4t\log(P_t\mathbb{1}_A)}$ has a LIPSCHITZ norm bounded by 1, uniformly in t. From this, HINO [27] deduces large deviations results when $t \to 0$ for the heat kernel measure.

Let us observe that when ρ is strictly positive, then one may let t goes to infinity in the inequalities above. Then, since $P_t f$ converges to $\int f\,d\mu$, one gets in this way the spectral gap inequality and the logarithmic SOBOLEV inequality for the invariant measure μ.

Proof. — (Of Proposition 3.3.) We shall just give a hint tof the proof, and we refer to [3] for more details.

The fact that any of these inequalities imply the $CD(\rho,\infty)$ inequality comes from a closer look at the behaviour in $t = 0$ of the inequality : it is just an asymptotic expansion using the fact that $P_t = \mathrm{Id} + tL + t^2L^2/2 + o(t^2)$ in $t = 0$. (One has to go to the second derivatives for the six last ones.)

For the converse implication, let us work on the local spectral gap inequality (4) for example. The only trick is to fix t and to focus on the function $H(s) = P_s((P_{t-s}f)^2)$, for $s \in [0,t]$. Then, we have

$$H'(s) = (\partial_s P_s)((P_{t-s}f)^2) + 2P_s(P_{t-s}f\,\partial_s P_{t-s}f).$$

Now, we know that $\partial_s P_s = LP_s = P_s L$, and therefore

$$H'(s) = 2P_s(\Gamma(P_{t-s}f, P_{t-s}f)),$$

from the definition of Γ.

If we take the second derivative, we get

$$H''(s) = 4P_s(\Gamma_2(P_{t-s}f, P_{t-s}f)),$$

for the same reason.

Therefore, the inequality $CD(\rho,\infty)$ tells us that $H''(s) \geq 2\rho H'(s)$, from which we get the inequality (2). Once we have this inequality, we get both upper and lower bounds on the derivative of H :

$$2\exp(2\rho(s))(\Gamma(P_tf,P_tf))\leq H'(s)\leq 2\exp(-2\rho(t-s))P_t\Gamma(f,f),$$

which gives (4) and (5).

The other ones are a bit more tricky, since we play the same game replacing H by $P_s(\Gamma(P_{t-s}f,P_{t-s}f)^{1/2})$. Then, instead of using $\Gamma_2\geq\rho\Gamma$, we have to use a stronger form, namely

$$4\Gamma(f,f)[\Gamma_2(f,f)-\rho\Gamma(f,f)]\geq\Gamma(\Gamma(f,f),\Gamma(f,f)).$$

This is where the diffusion assumption comes into play. With the use of the change of variable formula, one may prove that the $CD(\rho,\infty)$ inequality implies this last one, which is apparently stronger. In fact, in Riemannian geometry, the passage from the $CD(\rho,\infty)$ inequality to this stronger form just boils down to

$$|\nabla f|^2|\nabla\nabla f|^2\geq(\nabla\nabla f(\nabla f,\nabla f))^4,$$

which is quite obvious from the definitions (see [6]).

This gives the majorization (3). Then, if Φ is any smooth real function, and if $H(s)=P_s(\Phi(P_{t-s}f))$, one has as before

$$H'(s)=P_s[\Phi''(P_{t-s}f)\Gamma(P_{t-s}f,P_{t-s}f)].$$

If Φ is convex, one may use the bound

$$\Gamma(P_{t-s}f,P_{t-s}f)\leq[P_{t-s}(\Gamma(f,f)^{1/2})]^2\exp(-2\rho(t-s)),$$

then the fact that, since P_{t-s} is a Markov operator, one gets

$$\Phi''(P_{t-s}f)(P_{t-s}g)^2\leq P_{t-s}(g^2\Phi''(f)),$$

with $g=\sqrt{\Gamma(f,f)}$ as soon as $1/\Phi''$ is concave (using SCHWARZ' and then JENSEN's inequalities), and this gives the upper bounds. To get the lower bounds, we play the game in the other way, since for the same reason

$$P_s(\Phi''(h)\Gamma(g,g))\geq\Phi''(P_s(h))[P_s(\Gamma(g,g)^{1/2})]^2,$$

and then we use the lower bound on $P_s(\Gamma(g,g)^{1/2})$, with $g=P_{t-s}f$. ∎

Point (8) of Proposition 3.3 leads to the $CD(\rho,\infty)$ criterion for the logarithmic SOBOLEV inequality, for $\rho>0$, but is not enough for the $CD(0,\infty)$ criterion (3) of Theorem 3.2 which relies on an extra assumption on the distance function.

There is an additional tool, which is due to WANG [39]. In fact, if the distance function $d(x,y)$ is the infimum of the lenght of the curves joining x to y, then, using the same kind of arguments, WANG proved that under $CD(\rho,\infty)$, one has for any function f

$$(P_tf)^2(y)\leq P_t(f^2)(x)\exp[\rho(e^{2\rho t}-1)^{-1}d^2(x,y)].$$

(See [3]). This inequality is the key tool to prove that if the distance function is exponentially square integrable, and if $CD(0,\infty)$ holds, then the measure satisfies the logarithmic SOBOLEV inequality (see [3] for more details).

One should notice that these local inequalities do not at all require the reversibility of the measure μ. The measure μ appears only as the limit of P_t when t goes to infinity.

One may conjecture that the same holds true for the spectral gap inequality, replacing exponentially square integrable by exponentially integrable. This has been proved by S. Bobkov [17] in $\mathbb{R}^n$ with the usual Euclidean gradient, but with constants which depend on the dimension. But, contrary to what happens in this case, it is not enough in general for an operator satisfying $CD(0,\infty)$ to have a finite reversible measure to satisfy a spectral gap inequality. An infinite product of copies of $\mathbb{R}$, each equipped with an exponential measure $\exp(-a_i|x|)dx$, with a sequence of (a_i) going to 0, gives a counterexample.

There is no such simple tools to prove the Sobolev inequality starting from the $CD(\rho,n)$ inequality. The idea is first to prove an logarithmic entropy-energy inequality (20) by studying the decay of $E(t)=\int P_tf\log P_tf d\mu$ for the reversible measure μ. A precise analysis of the $CD(\rho,n)$ inequality leads to

$$\partial_t^2 E \geq -\rho\partial_t E + \frac{1}{n}(\partial_t E)^2,$$

which gives the logarithmic entropy-energy inequality (20) by integration. Then, one knows that the tight Sobolev inequality holds, and, by a compactness argument, that, for any $p<2n/(n-2)$ and any $a>1$, there is an inequality

$$\|f\|_p^2 \leq a\|f\|_2^2 + c(a,p)\mathcal{E}(f,f),$$

with an optimal $c(a,p)$ for which there exists an extremal non constant function f satisfying the equality (this would not be true at the critical exponent $p=2n/(n-2)$).

It remains then to do some manipulations on this function f (change it into f^α with a good choice of α, apply $\Gamma(f,\cdot)$ to it, then integrate), to obtain that

$$C(a,n) \leq \frac{n-1}{\rho}\frac{4}{n(n-2)}.$$

We may then take the limit as p converges to the critical value. (See [5] for more details).

If we want to understand the role of dimension, the spectral gap inequality may be a simple test. For example, under $CD(\rho,n)$, with $\rho>0$ it is quite easy to see that any non 0 eigenvalue λ of $-L$ must satisfy $\lambda\geq\rho\frac{n}{n-1}$, when the measure is reversible. Indeed, from the definitions of Γ_2 and Γ, we have

$$\langle\Gamma_2(f,f)\rangle = \langle(Lf)^2\rangle \text{ and } \langle\Gamma(f,f)\rangle = -\langle f,Lf\rangle$$

and then, if $Lf=-\lambda f$, then from $CD(\rho,n)$ one gets

$$\lambda^2\langle f^2\rangle \geq \rho\lambda\langle f^2\rangle + \frac{\lambda^2}{n}\langle f^2\rangle,$$

from which we may see that if $\lambda\neq 0$, then $\lambda\geq\rho n/(n-1)$.

On the other hand, letting the local spectral gap inequality go to infinity, we just get $\lambda\geq\rho$; this is not surprising since the local spectral gap inequality does not capture the dimension. But this bound does not require the symmetry assumption. We shall see in the next section some other local inequalities which capture the dimension and allow us to prove this bound on the spectral gap constant even in the non symmetric case.

Let us also mention that, in the symmetric case, one may indeed improve the previous results. In fact, under the $CD(\rho, n)$ inequality with $\rho > 0$, one may prove that the logarithmic SOBOLEV constant is bounded above by $\frac{2(n-1)}{\rho n}$, which is the critical value for the sphere. But the method of proof requires some arguments relying on symmetry that we cannot deduce from the local inequalities of Proposition 3.3.

To close this section, let us come back to the local spectral gap and logarithmic SOBOLEV inequality for the Gaussian measure and the usual Euclidean gradient. If we want to prove the logarithmic SOBOLEV inequality for the standard Gaussian measure on $\mathbb{R}$ or $\mathbb{R}^n$, we may use, as we saw, the local logarithmic SOBOLEV inequality for the ORNSTEIN-UHLENBECK semigroup, which satisfies $CD(1, \infty)$, and let t goes to infinity, using the fact that the Gaussian measure is the invariant (here reversible) measure for this semigroup. We may also use the standard Brownian motion in $\mathbb{R}^n$, which has the generator $\frac{1}{2}\Delta$ satisfying $CD(0, n)$ (and therefore $CD(0, \infty)$), and use the local inequality of Proposition 3.3 at time 1, since $P_1(0, dy)$ is then exactly the standard Gaussian measure.

On the other hand, assume that we already know the logarithmic SOBOLEV inequality (16) (or the spectral gap inequality (14)) for the standard Gaussian measure. The standard heat kernel in $\mathbb{R}^d$, with generator $\frac{1}{2}\Delta$, may be represented as

$$P_t f(x) = \int f(x + \sqrt{t} y) \gamma(dy),$$

where γ is the standard Gaussian measure. Therefore, using dilations and translations, we recover from the logarithmic SOBOLEV (or the spectral gap) inequality for γ the local logarithmic SOBOLEV (or spectral gap) inequalities for P_t.

We may play the same game with the ORNSTEIN-UHLENBECK semigroup, since it can be represented as

$$P_t f(x) = \int f(e^{-t} x + \sqrt{1 - e^{-2t}} y) \, \gamma(dy),$$

(MEHLER's formula), and therefore we may also recover the local logarithmic SOBOLEV and spectral gap inequalities for the ORNSTEIN-UHLENBECK semigroup from the corresponding inequalities for the measure γ using translations and dilations. Since these inequalities are all equivalent to the $CD(0, \infty)$ or $CD(1, \infty)$ inequalities for the standard heat kernel or ORNSTEIN-UHLENBECK operator in $\mathbb{R}^n$ when $t \to 0$, we may see those $CD(\rho, \infty)$ inequalities for these models as infinitesimal versions of logarithmic SOBOLEV or spectral gap inequalities.

It is not clear whether the same game may be played with the $CD(\rho, n)$ inequalities, and this is the question we shall address in Section 4.

In fact, most of the methods using the finite dimension information related to the $CD(\rho, n)$ inequalities rely either on integration by parts arguments (similar to the one we have just described for spectral gap), or on the use of maximum principle, which says that, for a differential operator like L, $LH \leq 0$ at any maximum of H. For example, one may deduce from this the celebrated LI-YAU inequality, which asserts, under $CD(0, n)$, if u is any positive solution of the heat equation $\partial_t u = Lu$, then if we set $f = \log u$, one has

$$\Gamma(f, f) - \partial_t f \leq \frac{n}{2t}.$$

The idea is to use the maximum principle at a maximum point of $t[\Gamma(f,f)-\partial_t f]$, and then use the $CD(0,n)$ assumption.

This inequality is an equality for the heat kernel in $\mathbb{R}^n$, and this result is therefore optimal. There is of course a similar result under the $CD(\rho,n)$ inequality, much more complicated to state. Let us mention that this kind of results leads to parabolic HARNACK inequalities of the form

$$\frac{u(t,x)}{u(t+s,y)} \leq (\frac{t+s}{t})^{n/2} \exp(\frac{d^2(x,y)}{4s})$$

(see [13] for extensions to the general case).

Also, using the same method of maximum principle, one may prove that, under $CD(\rho,n)$ and an upper bound δ on the diameter, the smallest non zero eigenvalue of $-L$ is bounded from below by the smallest non zero eigenvalue of the corresponding one-dimensional models, on the symmetric interval $[-\delta/2,\delta/2]$, with NEUMANN boundary conditions (see [14]).

4 Conformal invariance of Sobolev inequalities.

Most of the material presented here is taken from [36].

Let us rewrite the optimal SOBOLEV inequality of the sphere in the following form:

$$\|f\|_p^2 \leq \frac{4}{n(n-2)}[\mathcal{E}(f,f) + \frac{(n-2)}{4(n-1)}\int f^2 \mathrm{sc}(x)\, d\mu(x)],$$

where $\mathrm{sc}(x)$ denotes the scalar curvature of the sphere, that is, the trace of the RICCI curvature (here $n(n-1)$).

It turns out that, for a Riemannian manifold of dimension n, the inequality

$$\|f\|_p^2 \leq C[\mathcal{E}(f,f) + \frac{(n-2)}{4(n-1)}\int f^2 \mathrm{sc}(x)\, d\mu(x)],$$

is invariant under a conformal change of the metric. This means that if we pick any positive smooth function σ and change Γ into $\sigma^{-2}\Gamma$ and μ into $\sigma^n\mu$, the same inequality is true, with the scalar curvature of the new metric σg. It is essential here that $p = 2n/(n-2)$, and this property is restricted to LAPLACE-BELTRAMI operators.

Indeed, if we apply the intial inequality to $f = g\sigma^{(n-2)/2}$ and write

$$\mathcal{E}(gh,gh) = \int h^2\Gamma(g,g)\, d\mu - \int g^2 hL(h)\, d\mu,$$

we get exactly the inequality for σg with the new scalar curvature instead of the old one (it is absolutely crucial here that the constant in front of the term $\int f^2 \mathrm{sc}(x)\, d\mu$ is $\frac{(n-2)}{4(n-1)}$ and nothing else).

The new metric is just a conformal transform of the old one. This is why, up to the $\mathrm{sc}(x)$ term, all Riemannian manifolds which are conformal to each other share the same optimal SOBOLEV constant.

In particular, this is true for the sphere, the Euclidean space and the hyperbolic space. The stereographic projection is a conformal tranformation of the n-dimensional sphere onto $\mathbb{R}^n$. More precisely, if we write the spherical metric on $\mathbb{R}^n$ using this stereographic projection as a system of coordinates, we get

$$\Gamma_S(f,f) = (\frac{1+|x|^2}{2})^2 \Gamma_E(f,f),$$

where Γ_S and Γ_E denote respectively the spherical and Euclidean carrés du champ.

This is easy to understand if we remember that inversions are conformal transformations in the Euclidean space, and that the stereographic projection is the restriction to the sphere of the inversion in $\mathbb{R}^{n+1}$ which is centered on the pole of the stereographic projection.

It is also true that the hyperbolic space has a metric which is conformal with the Euclidean metric of $\mathbb{R}^n$ since, if we use the half space representation of the hyperbolic space $H_n = \mathbb{R}^{n-1} \times \mathbb{R}_+$, the hyperbolic carré du champ is

$$\Gamma_H(f,f) = y_n^2 \Gamma_E(f,f),$$

where y_n is the coordinate in $\mathbb{R}_+$. This explains the similar forms of the SOBOLEV inequality of the three models.

Now, we may try the same trick on the SOBOLEV inequality of the sphere that we used for the logarithmic SOBOLEV inequality of the Gaussian measure. On the $\mathbb{R}^n$ representation on the sphere, we may apply translations and dilations. This amounts to use conformal transformations on the sphere to deform the SOBOLEV inequality. In order to describe what we get, we need some additional notations.

Let $Q_t^m(x,dy)$ the MARKOV kernel defined on $\mathbb{R}^n$ as follows

$$Q_t^m(x,dy) = \frac{t^m}{(t^2+|x-y|^2)^{(m+n)/2}} \frac{dy}{c_{m,n}},$$

where $m>0$, dy denotes the LEBESGUE measure on $\mathbb{R}^n$ and $c_{m,n}$ is a normalizing constant which depends only on m and n.

Then, the SOBOLEV inequality on the sphere becomes, with $p = 2n/(n-2)$,

$$Q_t^n(f^p)^{2/p} \leq Q_t^n(f^2) + \frac{4(n-1)}{n(n-2)(n-4)} t^2 Q_t^{(n-4)}(\Gamma_E(f,f)).$$

Notice that the family Q_t^m is not a semigroup. But we may nevertheless still let t go to 0. Unfortunately, this just gives in $t=0$ that $\Gamma_E(f,f) \geq 0$. For the logarithmic SOBOLEV inequality in the Gaussian case, the identity was precise enough near 0 for the first order term to vanish, and give the $CD(\rho,\infty)$ inequality for the second order terms. In the language of [36], the first one is not $CD(\rho,n)$-sharp, while the second is $CD(\rho,\infty)$-sharp.

We may play the same game with the spectral gap inequality, which gives

$$Q_t^n(f^2) \leq Q_t^n(f)^2 + t^2 \frac{n-1}{n(n-4)} Q_t^{n-4}(\Gamma_E(f,f)),$$

and we do not get a better result when t goes to 0.

But we may change a little the spectral gap inequality of the sphere in the following way : let h be the function $(1+|x|^2)/2$ in $\mathbb{R}^n$. Then, the spherical LAPLACE-BELTRAMI operator is

$$\Delta_S = h^2\Delta - (n-2)h\nabla h\nabla,$$

where Δ is the Euclidean Laplacian, since it has carré du champ $h^2\Gamma_E$ and reversible measure $h^{-n}dx$. Let $\hat{\Delta}_S$ be the operator

$$\hat{\Delta}_S = h\Delta - n\nabla h\nabla.$$

It has carré du champ $h\Gamma_E$ and reversible measure $h^{-(n+1)}dx$. We may observe that $\hat{\Delta}_S(h) = -n(h-1)$, or symmetrically that $\Delta_S h^{-1} = -n(h^{-1}-1)$ (these two properties are equivalent).

One may then observe that

$$(\hat{\Delta}_S + n\mathrm{Id})(hf) = (\Delta_S + n\mathrm{Id})f.$$

Since the spectral gap inequality for Δ_S may be written as

$$\langle(\Delta_S + n\mathrm{Id})f, f\rangle \leq \langle f\rangle^2,$$

for the reversible measure of Δ_S, then it is quite obvious that this spectral gap inequality is equivalent to the same spectral gap inequality for the operator $\hat{\Delta}_S$. (This has nothing specific to do with the spherical Laplacian, it is just a manipulation around an eigenvector of Δ_S associated with the lowest non zero eigenvalue.)

Then, we may again deform this new spectral gap inequality through dilations and translations, and we get

$$Q_t^{n+2}(f^2) \leq Q_t^{n+2}(f)^2 + \frac{t^2}{n}Q_t^n(\Gamma_E(f,f)). \tag{24}$$

It turns out that, when t goes to 0, this inequality is sharper than the previous one and gives

$$|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta_E f)^2.$$

Therefore, this new spectral gap inequality captures the $CD(0,n)$ inequality of $\mathbb{R}^n$: it is $CD(0,n)$-sharp in $\mathbb{R}^n$.

Our aim in what follows is to generalize such a result to get a characterization of $CD(0,n)$ inequality similar to the local spectral gap inequality (4).

For that, we first observe that, y being fixed, the density $q_t^m(x,y)$ of the kernel Q_t^m satisfies the following equation

$$[\Delta_x + \partial_t^2 - \frac{m-1}{t}\partial_t]q_t^m(x,y) = 0.$$

Define the operator L^m on $\mathbb{R}^n \times \mathbb{R}_+$ by

$$L^m = \Delta + \partial_t^2 - \frac{m-1}{t},$$

where Δ is the usual Euclidean Laplacian. Since $Q_t^m(x,dy)$ converges to the DIRAC mass at x when t goes to 0, we therefore see that $F(x,t) = Q_t^m(f)(x)$ is the solution on $\mathbb{R}^n \times \mathbb{R}_+$ of $L^m F = 0$ with $F(x,0) = f(x)$, this solution being the unique bounded solution on $\mathbb{R}^n \times \mathbb{R}_+$ when the boundary value $F(x,0)$ is bounded.

It perhaps looks surprising that for $m = n$, the operator $t^2 L^m$ is the hyperbolic Laplace-Beltrami operator in $\mathbb{R}^n \times \mathbb{R}_+$. This comes from the fact that dilations and translations on $\mathbb{R}^n$ are the images by the stereographic projection of conformal transformations on the sphere : up to rotations, these conformal transformations are restrictions to the sphere of inversions in $\mathbb{R}^{n+1}$ which leave the sphere invariant, and the action of these inversions in the interior of the unit ball are isometries for the hyperbolic structure of the ball. The parametrization chosen for these transformations is such that it respects this structure and explains the fact that the function $F(x,t)$ is harmonic for the hyperbolic structure in the half-space. In the end, the operator Q_t^n is nothing else than the Poisson kernel for the half space, when the Euclidean structure is replaced by the hyperbolic one.

It is quite easy to see from the definition of q_t^m that, if we denote by $p_t(x,y)$ the standard heat kernel in $\mathbb{R}^n$, which is

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp(\frac{-|x-y|^2}{4t}),$$

then we have

$$q_t^m(x,y) = \int_0^\infty p_s(x,y)\mu_t^m(ds),$$

where

$$\mu_t^m(ds) = t^m \exp(-t^2/4s)\frac{ds}{\alpha_m s^{1+m/2}} \tag{25}$$

is a probability measure on $[0,\infty)$, α_m beeing a normalizing constant.

Therefore, we have the representation

$$Q_t^m = \int_0^\infty P_s \mu_t^m(ds),$$

and this formula is the key of the general extension of the previous result.

Theorem 4.1 *[36]*

Assume that L is a diffusion operator, and let P_t be the associated heat kernel. Let

$$Q_t^m = \int_0^\infty P_s \mu_t^m(ds),$$

where $\mu_t^m(ds)$ is defined by equation (25).

Then, the following are equivalent :

1. *L satisfies the $CD(0,n)$ inequality;*
2. *For any $t > 0$, and any $f \in \mathcal{A}$, one has*

$$Q_t^{n+2}(f^2) \le Q_t^{n+2}(f)^2 + \frac{t^2}{n} Q_t^n(\Gamma(f,f)).$$

We also have a similar version related to the $CD(\rho, n)$ inequality with $\rho > 0$, but it is less explicit. This version comes from the same considerations on the MARKOV kernels Q_t^m, but we have to look at them from another point of view. To understand this, we come back to the construction of the kernels Q_t^m on $\mathbb{R}^n$. One may carry everything back onto the sphere through stereographic projection, and then, we may see the MARKOV kernel $Q_t^m(x, dy)$ as families of probability measures on the sphere, indexed by a point z in the unit ball of $\mathbb{R}^{n+1}$, the point z being the image of the point (x, t) through the inversion in $\mathbb{R}^{n+1}$ which maps the unit sphere onto the hyperplane $\mathbb{R}^n$. Then, we may parametrize z as (r, y), with $r \in [0, 1]$ and $y = z/|z| \in S_n$. We get in such a way a family of MARKOV kernels S_r^m on the sphere which are parametrized by $r \in [0, 1]$. We just did the analogous of the two parametrizations of the deformations of Gaussian measure by dilations and translations, the first one describing the Euclidean heat kernel and the second one the ORNSTEIN-UHLENBECK semigroup.

Writing the spectral gap inequality (24) for this parametrization, and letting the point (x, r) of the unit ball go to the boundary, we see that this inequality is sharp and gives the $CD(n-1, n)$ inequality of the sphere.

But this construction is hard to be carried to any general operator satisfying a $CD(\rho, n)$ inequality with $\rho > 0$, because these MARKOV kernels S_r^m may not be constructed only from the spherical Laplacian Δ_{S^n}. They are constructed in $\mathbb{R}^n$ through the choice of a function (an eigenvector) of the LAPLACE-BELTRAMI operator of the sphere, and this choice amounts to the choice of a point on the sphere. The stereographical projection, which carries the spherical structure onto the Euclidean one, also depends on the choice of the point on the sphere. Here, these two choices coincide, and this makes the whole machinery work.

To overcome this difficulty, SCHEFFER [36] followed another route, which we describe here only for the case of the spectral gap inequality. We have

Theorem 4.2 *[36] Let $\nu_t^n(ds)$ and $\hat{\nu}_t^n(ds)$ be the laws of the hitting time of 0 for the processes on $[0, \infty)$ with generators respectively*

$$L_n := \partial_t^2 - (n-1)\tanh(t)\partial_t,$$

and

$$\hat{L}_n := \partial_t^2 - \left((n-1)\tanh(t) + \frac{n+1}{\sinh(t)\cosh(t)}\right)\partial_t,$$

with an initial value $t > 0$.

Let P_t be a diffusion semigroup with generator L and let

$$Q_t^{n,+} = \int_0^\infty P_s \nu_t^n(ds) \; ; \; \hat{Q}_t^{n,+} = \int_0^\infty P_s \hat{\nu}_t^n(ds) \; .$$

Then, the following are equivalent :

1. *L satisfies a $CD(n-1, n)$ inequality.*

2. *For any $t > 0$, one has*

$$\hat{Q}_t^{n,+}(f^2) \leq \hat{Q}_t^{n,+}(f)^2 + \frac{1}{n}\tanh^2(t) Q_t^{n,+}(\Gamma(f, f)).$$

There is an obvious modification using any $CD(\rho, n)$ inequality for $\rho > 0$, but no version of this result is valid (for the moment) with $\rho < 0$.
Proof. — We only sketch the results for the $CD(0, n)$ case, the $CD(\rho, n)$ being similar. (See [36] for the general case.) The proofs are not really difficult, but require a bit of technicality. The fact that the local spectral gap inequalities imply the $CD(\rho, n)$ inequalities is just a simple analysis of the behaviour of the kernels near $t = 0$. For example, one has

$$Q_t^m(f) = \mathrm{Id} + \frac{t^2}{2(m-2)} Lf + \frac{t^4}{8m(m-2)(m-4)} L^2(f) + o(t^4), \; t \to 0^+$$

and an asymptotic expansion of the local spectral gap inequality up to t^4 gives the $CD(0, n)$ inequality (the fact that the terms in t^2 cancel is exactly the $CD(0, n)$-sharpness).

The converse comes from a sub-harmonicity lemma. From the contruction of Q_t^m, one may see that in the general case the function $F(x, t) = Q_t^{n+2}(f)(x)$ is a solution on $E \times \mathbb{R}_+$ of $\hat{L}_{n+2} F = 0$, where

$$\tilde{L}_{n+2} = L + \partial_t^2 - \frac{n+1}{t} \partial_t.$$

Set $\tilde{\Gamma}(F, F) = \Gamma_x(F, F) + (\partial_t f)^2$, and assume that $CD(0, n)$ holds. Then, if $\hat{L}_{n+2}(F) = 0$, we also have

$$\tilde{L}_{n+2}(\tilde{\Gamma}(F, F)) \geq 0. \tag{26}$$

Now, if (X_t, U_t) is the Markov process in $E \times \mathbb{R}_+$ generated by $\tilde{L}_{n+2}$ (the two coordinates are independent), and if S is the hitting time of 0 for the process U, one has, since $\tilde{L}_{n+2} F = 0$,

$$F(x, t) = E_{x,t}(f(X_S)) = Q_t^{n+2}(f)(x),$$

and

$$E_{x,t}[f^2(X_S)] = F^2(x, t) + E_{x,t}[\int_0^S \hat{\Gamma}(F, F)(X_s, U_s) ds].$$

From the subharmonicity lemma (26), one concludes that $\tilde{\Gamma}(F, F)(X_s, U_s)$ is a sub-martingale, and the last term is bounded from above by

$$2E_{x,t}[S\hat{\Gamma}(F, F)(X_S, U_S)] = 2E_{x,t}[S\Gamma(f, f)(X_S)].$$

Then, we get

$$Q_t^{n+2}(f)^2 \leq Q_t^{n+2}(f)^2 + 2E_{x,t}[S\Gamma(f, f)(X_S)].$$

It remains to identify the last term as $\frac{t^2}{n} Q_t^n(\Gamma(f, f))$, wich comes from considerations on the operator $\partial_t^2 - \frac{m-1}{t} \partial_t$. ∎

The reason why it is this operator which gives the sharp spectral gap inequality (and how one can device the operator in the positive curvature case) comes from quite subtle considerations on quasi-laplacians. The operators constructed above in the case of spheres are quasi-laplacians on $\mathbb{R}^n \times \mathbb{R}_+$, or on the ball, with constant curvatures (see [36] for more details).

There exist equivalent forms of these local inequalities with

$$Q_t(f^2) \le Q_t(f^p)^{2/p} + C(t,p)R_t(\Gamma(f,f))$$

for $p \in [1,2)$ which are $CD(\rho,n)$-sharp, but up to now no such inequality holds for SOBOLEV type inequalities (i.e. when $p \ge 2$).

As a consequence of these local inequalities, one may see for example that the $CD(\rho,n)$ inequality implies the spectral gap inequality with constant the $C = \rho n/(n-1)$, even in the absence of reversibility for the measure μ, by letting t go to infinity in the local spectral gap inequality (24).

5 Porous media equations and Sobolev inequalities.

Up to now, we were just able to prove the SOBOLEV inequality under the $CD(\rho,n)$ assumption, with $\rho > 0$. In this section, we want to investigate some SOBOLEV type consequences of the $CD(0,n)$ inequality. For this, we start from the Euclidean case, and rewrite the SOBOLEV inequality in a different way, following [21]. We begin with a simple remark, which is just a new presentation of the optimal constant in the Euclidean SOBOLEV inequality.

The extremal functions of the SOBOLEV inequality in $\mathbb{R}^n$ are

$$G_{\sigma,b}(y) = (\sigma^2 + b|x-y|^2)^{-(n-2)/2},$$

with $b > 0$, which satisfy

$$\Delta G_{\sigma,b} = -n(n-2)b\sigma^2 G_{\sigma,b}^{p-1},$$

where $p = 2n/(n-2)$ is the SOBOLEV exponent.

Let α_n be the best constant in the SOBOLEV inequality in $\mathbb{R}^n$, which was computed in the previous section by means of conformal transformations. Multiplying the previous identity by G and integrating, we get

$$\int |\nabla G_{\sigma,b}|^2\, dx = n(n-2)b\sigma^2 \int G_{\sigma,b}^p\, dx.$$

On the other hand, since we know that it is an extremal of the SOBOLEV inequality, we also get

$$\alpha_n \int |\nabla G_{\sigma,b}|^2\, dx = (\int G_{\sigma,b}^p\, dx)^{1-2/n}.$$

Comparing the two identities, setting $h_{\sigma,b} = \sigma^2 + b|x|^2$, we get

$$(\int h_{\sigma,b}^{-n}\, dx)^{-2/n} = \alpha_n n(n-2)b\sigma^2. \tag{27}$$

Also, if we notice that

$$|\nabla G_{\sigma,b}|^2 = b(n-2)^2 h_{\sigma,b}^{-n}(h_{\sigma,b} - \sigma^2),$$

we get

$$\int h_{\sigma,b}^{1-n}\,dx = 2\sigma^2\frac{n-1}{n-2}\int h_{\sigma,b}^{-n}\,dx. \tag{28}$$

These considerations allowed M. Del Pino and J. Dolbeault in [21] to state a different form of a Sobolev inequality in $\mathbb{R}^n$, that we shall describe. Let

$$H(x) = -x^{1-1/n} \text{ and } \Psi(x) = H'(x) = -\frac{n-1}{n}x^{-1/n}.$$

Let $b > 0$ be fixed and $v_\sigma = h_{\sigma,b}^{-n}$ (the authors restrict themselves to $b = 1/[2(n-1)]$ but this plays no role in what follows). Let f be a positive integrable function and let σ be such that $\int f\,dx = \int v_\sigma\,dx$. Then, one has

$$\int [H(f) - H(v_\sigma) - (f - v_\sigma)\Psi(v_\sigma)]\,dx \le \frac{n}{n-1}\frac{1}{4b}\int f|\nabla(\Psi(f) - \Psi(v_\sigma))|^2\,dx. \tag{29}$$

This inequality is used in their paper to control the convergence to equilibrium of some non-linear evolution equation that we shall describe later on.

To understand this inequality, let us expand the RHS. We write h for $h_{\sigma,b}$. We have

$$\begin{aligned}\int f|\nabla(\Psi(f) - \Psi(v_\sigma))|^2\,dx &= (\frac{n-1}{n})^2\int\frac{4}{(n-2)^2}|\nabla f^{(n-2)/(2n)}|^2\,dx\\ &\quad + \int\left[-\frac{2}{n-1}f^{1-1/n}\Delta h + f|\nabla h|^2\right]dx.\end{aligned}$$

Now, we know that $\Delta h = 2nb$ and $|\nabla h|^2 = 4b(h - \sigma^2)$.

Comparing the two sides, it just remains

$$\sigma^2\int f\,dx + \frac{1}{n-1}\int h^{1-n}\,dx \le \frac{1}{4b}\int |\nabla f^{(n-2)/(2n)}|^2\,dx.$$

Take the value of σ^2 given by equation (27) and the value of $\int h^{1-n}\,dx$ given by equation (28), and recall that $\int f\,dx = \int h^{-n}\,dx$ (this is the definition of σ). We get

$$(\int f\,dx)^{1-2/n} \le \alpha_n\int |\nabla f^{(n-2)/(2n)}|^2\,dx,$$

which is the optimal Sobolev inequality for $\mathbb{R}^n$.

As we already mentioned, the inequality (29) is used to control a non linear evolution equation, which is a modified version of the porous media equation :

$$\frac{\partial u}{\partial t} = -\nabla^* u[\nabla(\Psi(u) - \Psi(v_\sigma))] \tag{30}$$

with initial value $u(x,0) = f(x)$ (here ∇^* is the adjoint of the ∇ operator in $L^2(dx)$, that is the divergence operator). We shall not address here existence and regularity properties of the solutions of these equations in a general setting, but we shall show that this equation plays, with respect to the modified Sobolev inequality (29), the same role as the Ornstein-Uhlenbeck evolution equation for the logarithmic

Sobolev inequality for the Gaussian measure, where the $CD(0,\infty)$ inequality is replaced by the $CD(0,n)$ inequality.

Let us assume then that we have an operator L satisfying the $CD(0,n)$ inequality, with a reversible measure μ, and let ∇^* be the adjoint of ∇ in $L^2(\mu)$, in which case $L=-\nabla^*\nabla$.

We take as before the function $H(x)=-x^{1-1/n}$ and $\Psi(x)=H'(x)$. Let v be a positive function such that $-\Psi(v)$ is strictly convex : $-\nabla\nabla\Psi(v)\geq \rho g$, where $\rho>0$ and g denotes the metric associated to L, the computation of the second derivative $\nabla\nabla\Psi(v)$ being made for this metric. This function v is fixed in what follows. Let us assume that $u(x,t)$ is a smooth positive solution of

$$\frac{\partial u}{\partial t}=-\nabla^*[u\nabla(\Psi(u)-\Psi(v))], \tag{31}$$

with $u(x,0)=f(x)$. We shall assume that u has at least two derivatives, locally majorized by an integrable function, to justify all the integrations by parts made below, and we assume that $\int f\,d\mu=\int v\,d\mu$. Since the function v in the equation (31) is defined up to some constant added to $\Psi(v)$, this choice corresponds to the choice of the parameter σ described previously. Then we have

Theorem 5.1 *Assume that $CD(0,n)$ holds and $-\nabla\nabla\Psi(v)\geq\rho\mathrm{Id}$. Then, u converges to v when t goes to infinity (in a sense described below) and we have*

$$\int[H(f)-H(v)-(f-v)\Psi(v)]\,d\mu\leq\frac{1}{2\rho}\int f|\nabla(\Psi(f)-\Psi(v))|^2\,d\mu, \tag{32}$$

where $\int f\,d\mu=\int v\,d\mu$.

One should notice that, when $n=\infty$, one could replace $\Psi(x)$ by $\log(x)$, and the equation (31) then becomes

$$\partial_t u=Lu-\nabla u\nabla\log(v)-uL(\log(v)), \tag{33}$$

and if we do the change of variables $u=wv^{-1}$, the equation becomes

$$\partial_t w=Lw+\nabla w\nabla\log v, \tag{34}$$

which is the heat equation related to the operator $L_v=L+\nabla\log v\nabla$, which has reversible measure $v\,d\mu$. Equation (33) is the Fokker-Planck equation, which describes the evolution of the density of the associated Markov process associated with L_v, with respect to the initial measure μ, while equation (34) is the heat equation which describes the evolution of the same density, but with respect to the invariant measure $vd\mu$.

In this case, the inequality (32) gives in fact the logarithmic Sobolev inequality related to L_v, which satisfies $CD(\rho,\infty)$ as soon as L satisfies $CD(0,\infty)$ and $-\nabla\nabla\log(v)\geq\rho g$. If we chose $v=\exp(-|x|^2/2)$, we get the usual logarithmic Sobolev inequality with respect of the Gaussian measure.

This result is therefore nothing else that an extension to the finite dimensional case of the classical logarithmic Sobolev inequality under the $CD(\rho,\infty)$ criterion, and the method is completely similar, except that we have moved for convenience from the heat equation to the Fokker-Planck one.

Proof. — We shall not be very formal in what follows, but we shall show the main ideas, without justifying all the derivations and integrations by parts which require some precise analysis of the solutions of equation (31).

First, following [21], we define the entropy related to the modified porous media equation (MPME for short) to be

$$E(f) = \int [H(f) - f\Psi(v)]\, d\mu.$$

Since the function H is convex, we have $H(f) - H(v) \geq (f - v)\Psi(v)$, therefore $E(f) - E(v) \geq 0$ and v is the unique minimum of E.

Now, if we set $\Phi(t) = E(u_t)$, where u is a solution of the MPME (31), then we get by integration by parts

$$\partial_t \Phi = \int \partial_t u(\Psi(u) - \Psi(v))\, d\mu. \tag{35}$$

Now, integration by parts show that, for any smooth function K, one has

$$\int \partial_t u K d\mu = -\int u\Gamma(\xi, K) d\mu, \tag{36}$$

where ξ is the function $\Psi(u) - \Psi(v)$. This has nothing specific with the precise form of the function Ψ.

If we apply this to equation (35), we get

$$partial_t \Phi = -\int u\Gamma(\xi, \xi) d\mu. \tag{37}$$

For any smooth function f, let

$$I(f) = \int f|\nabla(\Psi(f) - \Psi(v))|^2\, d\mu.$$

Our aim here, is to prove that

$$I(u_t) \leq \exp(-2\rho t) I(u_0).$$

If this holds, then $I(u)$ converges to 0 as t goes to infinity. We then see that $\nabla\Psi(u)$ converges to $\nabla\Psi(v)$. On the other hand, we have that $\partial_t \int u\, d\mu = 0$, therefore $\int u\, d\mu = \int v\, d\mu$, and then u converges to v.

$$E(f) - E(v) = \int_0^\infty I(u_s)\, ds \leq \frac{1}{2\rho} I(u_0),$$

which is the inequality (32).

Now, the trick is to compute $\partial_t I(u(t))$, and we write $I(t) = I(u(t))$ for simplicity. To do that, since there are many different forms under which we may write the same expression, we use our operators Γ and Γ_2 to get a canonical form. We have

Lemma 5.2 *Let* $S = \Psi(u)$, $\xi = \Psi(u) - \Psi(v)$, *and* $R(x) = x\Psi''(x)/\Psi'(x)$. *Then,*

$$\partial_t I(u) = -\int u K d\mu,$$

with

$$\begin{aligned} K &= 2u\Psi'(u)\Gamma_2(\xi,\xi) + \Gamma(\xi,\Gamma(\xi,\xi)) + (R(u)+2)\Gamma(S,\Gamma(\xi,\xi)) \\ &\quad + 2R(u)\Gamma(\xi,\Gamma(\xi,S)) + 2\frac{R(u)+1+uR'(u)}{u\Psi'(u)}\Gamma(\xi,S)^2. \end{aligned}$$

Proof. — In this formula, the function Ψ is not required to be the function $-\frac{n-1}{n}x^{-1/n}$, and the result is general.

Since $\partial_t \xi = \partial_t u \Psi'(u)$ we have

$$\partial_t I(u) = \int [\partial_t u \Gamma(\xi,\xi) + 2u\Gamma(\partial_t u\Psi'(u),\xi)] d\mu. \tag{38}$$

Then, we deal with the second term noticing that, for any reasonable functions (u, f, g), we have

$$\int u\Gamma(f,g) d\mu = -\int f[uL(g) + \Gamma(u,g)] d\mu,$$

which comes from the definition of Γ and the fact that for any pair (h,k) of functions, one has

$$\int hL(k) d\mu = -\int \Gamma(h,k) d\mu.$$

Then, the second term in (38) may be written as

$$-2\int \partial_t u \Psi'(u)[uL(\xi) + \Gamma(u,\xi)] d\mu.$$

We then apply the integration by parts formula (36) to get

$$-\partial_t I(u) = \int u R_1 d\mu,$$

with

$$R_1 = \Gamma(\xi,\Gamma(\xi,\xi)) - 2\Gamma(\xi,\Gamma(S,\xi)) - 2\Gamma(\xi, u\Psi'(u)L\xi).$$

We now deal with the third term of the last formula.

$$\begin{aligned} \Gamma(\xi, u\Psi'(u)L\xi) &= u\Psi'(u)\Gamma(\xi, L\xi) + L(\xi)(u\Psi''(u) + \Psi'(u))\Gamma(\xi,u) \\ &= u\Psi'(u)\Gamma(\xi, L\xi) + L(\xi)(R(u)+1)\Gamma(\xi,S). \end{aligned}$$

Now, from the definition of Γ_2 we have

$$2\Gamma(\xi, L\xi) = L(\Gamma(\xi,\xi)) - 2\Gamma_2(\xi,\xi),$$

and

$$\int u^2\Psi'(u)L(\Gamma(\xi,\xi)) d\mu = -\int \Gamma(u^2\Psi'(u),\Gamma(\xi,\xi)) d\mu = \int u(R(u)+2)\Gamma(S,\Gamma(\xi,\xi)) d\mu.$$

The term
$$\int uL(\xi)(R(u)+1)\Gamma(\xi,S)d\mu$$
is itself integrated by parts to give
$$-\int \Gamma(\xi, u(R(u)+1)\Gamma(\xi,S))d\mu =$$
$$-\int u(R(u)+1)\Gamma(\xi,\Gamma(\xi,S))d\mu - \int \frac{R(u)+1+uR'(u)}{\Psi'(u)}\Gamma(\xi,S)^2 d\mu.$$

It remains to collect all the terms to get the result.

∎

The modified porous media equation is the case when $\Psi(x) = -\frac{n-1}{n}x^{-1/n}$, $H(x) = x^{1-1/n}$. In this case $R = -1-1/n$.

In this particular situation, the expression of $K := K_n$ is much simpler and we get

$$\begin{aligned} K_n &= -(2/n)S\Gamma_2(\xi,\xi) + \Gamma(\xi,\Gamma(\xi,\xi)) \\ &\quad +(1-1/n)\Gamma(S,\Gamma(\xi,\xi)) - 2(1+1/n)\Gamma(\xi,\Gamma(\xi,S)) \\ &\quad +2\frac{\Gamma(\xi,S)^2}{S} \end{aligned} \tag{39}$$

∎

With this expression, we are ready to use our curvature-dimension inequality assumption on the generator L.

We first begin with a simple computation.

Lemma 5.3 *Let w and ξ be two smooth function. Then,*
$$\int w(L\xi)^2 d\mu = \int [w\Gamma_2(\xi,\xi) + \frac{1}{2}\Gamma(w,\Gamma(\xi,\xi)) + \Gamma(\xi,\Gamma(\xi,w))]d\mu.$$

Proof. —

This comes from successive integration by parts, since
$$\begin{aligned} \int w(L\xi)^2\, d\mu &= -\int \Gamma(\xi, wL\xi)\, d\mu \\ &= -\int [w\Gamma(\xi,L\xi) + L\xi\Gamma(\xi,w)]\, d\mu \\ &= \int [w\Gamma_2(\xi,\xi) - \frac{1}{2}wL\Gamma(\xi,\xi) + \Gamma(\xi,\Gamma(\xi,w))]\, d\mu \\ &= \int w\Gamma_2(\xi,\xi)\, d\mu + \frac{1}{2}\int \Gamma(w,\Gamma(\xi,\xi))\, d\mu + \int \Gamma(\xi,\Gamma(\xi,w))\, d\mu. \end{aligned}$$

∎

From lemma 5.3, we get

Proposition 5.4 *Assume that the $CD(0,n)$ assumption holds for L with some $n>1$. Then, for any two smooth functions w and ξ, with $w \geq 0$, we have*
$$\begin{aligned} \int w\Gamma_2(\xi,\xi)d\mu &\geq \\ &\frac{1}{(n-1)}\int [\frac{1}{2}\Gamma(w,\Gamma(\xi,\xi)) + \Gamma(\xi,\Gamma(w,\xi))]d\mu \end{aligned}$$

Now, in the case of interest, we have $w = 2(n-1)u^{(n-1)/n}/n^2$, and $S = -(n-1)u^{-1/n}/n$. Then we have

$$\Gamma(w, \Gamma(\xi, \xi)) = 2\frac{n-1}{n} u\Gamma(S, \Gamma(\xi, \xi)),$$

whereas

$$\Gamma(\xi, \Gamma(w, \xi)) = 2\frac{n-1}{n} u\Gamma(\xi, \Gamma(S, \xi)) - 2(n-1)u\frac{\Gamma(\xi, S)^2}{S}.$$

Comparing the expressions given by the formula (39) and Lemma 5.4, we get in the end a very simple expression : if $CD(0,n)$ holds, then,

$$-\partial_t I(u) \geq \int uK \, d\mu,$$

where

$$K = \Gamma(\xi + S, \Gamma(\xi, \xi)) - 2\Gamma(\xi, \Gamma(S, \xi)).$$

Now, a simple computation in Riemannian geometry shows that

$$K = 2\nabla\nabla(\xi - S)(\nabla\xi, \nabla\xi).$$

Here, $\xi - S = \Psi(v)$, and the hypothesis on v just tells us that $K \geq 2\rho\Gamma(\xi, \xi)$, which in turns tells us that

$$-\partial_t I(t) \geq 2\rho I(t),$$

and therefore $I(t) \leq \exp(-2\rho t)I(0)$.

Recently, J.Demange [22] got a similar result under the $CD(\rho, n)$ assumption, with a function v which satisfies the assumption

$$-\nabla\nabla\Psi(v) \geq \pm\Psi(v)g,$$

the sign depending of the sign of ρ, which extends in the same way the optimal Sobolev inequality on spheres and hyperbolic spaces.

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